

LIE SYMMETRIES, CONSERVATION LAWS AND EXACT SOLUTIONS FOR A CLASS OF MULTIDIMENSIONAL KDV-TYPE EQUATIONS

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Abstract

This work presents a detailed Lie symmetry investigation and develops conservation laws for three multidimensional KdV-type nonlinear evolution equations in (1+1), (1+2), and (1+3) dimensions. Using the multiplier approach together with the double-reduction method, several new invariant solutions and associated conserved vectors are obtained without relying on any Lagrangian formulation. The resulting symmetries are systematically linked with their corresponding conservation laws, illustrating how invariant transformations relate to physical conservation principles. The proposed reduction strategy yields implicit and explicit exact solutions that generalize known one-dimensional mKdV models to higher-dimensional nonlinear systems. The proposed unified framework provides a systematic approach for analyzing multidimensional KdV-type equations and yields new invariant solutions that enrich the existing analytical results for nonlinear evolution equations.

1. Introduction

Partial differential equations (PDEs) play a fundamental role in describing a wide variety of physical phenomena, including fluid dynamics, plasma physics, nonlinear optics, and wave propagation. Obtaining exact analytical solutions of nonlinear PDEs remains one of the central challenges in applied mathematics, since such solutions provide valuable insight into the qualitative behaviour and underlying mathematical structure of complex systems. Among the available analytical techniques, symmetry-based methods have become powerful tools for reducing nonlinear models, constructing exact solutions, and revealing important structural properties of differential equations. [3, 7]

Lie symmetry analysis has become one of the most widely used analytical techniques for investigating nonlinear evolution equations. By exploiting the invariance properties of differential equations, this method enables the systematic construction of similarity reductions, invariant solutions and symmetry classifications. Owing to its effectiveness, Lie symmetry analysis has been successfully applied to numerous nonlinear models, particularly KdV-type equations, where it has provided valuable insight into the mathematical structure and exact behaviour of nonlinear wave phenomena. [4, 6] Conservation laws constitute one of the fundamental analytical tools for investigating nonlinear partial differential equations, as they provide important information about the physical behaviour, stability and integrability of

mathematical models. In addition to their theoretical significance, conserved quantities are widely used to verify analytical and numerical solutions. Several systematic approaches have been developed for constructing conservation laws, among which the direct multiplier method has gained considerable attention because it can be applied even when a variational formulation is unavailable. This flexibility makes the multiplier approach particularly suitable for a broad class of nonlinear evolution equations. [1, 2, 9]

The multiplier method provides an efficient framework for deriving conserved vectors directly from governing differential equations without requiring a Lagrangian structure. When these conservation laws are associated with Lie symmetries, the double-reduction theory offers a systematic procedure for reducing nonlinear partial differential equations and constructing exact invariant solutions. The combination of these techniques has proved to be an effective analytical strategy for studying nonlinear evolution equations of increasing complexity. [5] Recent investigations have continued to explore the analytical structure of KdV- and KP-type nonlinear evolution equations through symmetry methods, conservation laws, and exact-solution techniques. Wazwaz et al. [8] studied extensions of KdV and modified KdV equations, while Zhao et al. [10] investigated Lie symmetries and conservation laws for a (2+1)-dimensional KP-type model. Related symmetry-based analyses have also been reported for generalized and higher-dimensional KdV/KP

equations [11, 12, 13], whereas recent work on the mKdV equation has further highlighted the role of symmetry structures in constructing nonlinear wave solutions [14]. These contributions demonstrate the continuing importance of symmetry-based techniques for KdV-type equations; however, the models are generally treated separately and by different analytical procedures. In the present study, the nKdV, mKdV-KP, and generalized KdV-like equations are therefore considered within a common analytical setting. Lie point symmetries are combined with multiplier-derived conservation laws and double reduction to obtain invariant reductions and exact solutions across (1+1), (1+2), and (1+3)-dimensional models.

The remainder of this paper is organized as follows. Section 2 presents the fundamental mathematical definitions and operators used throughout the analysis. Sections 3–5 contain the Lie symmetry analysis, conservation laws, double-reduction procedure, and exact solutions for the three multidimensional KdV-type equations considered in this study. Finally, Section 6 summarizes the principal findings and discusses the significance of the obtained results.

2. Fundamental Operators

This section summarizes the key mathematical definitions and operators employed in the symmetry and conservation-law analysis. For completeness, the principal formulations of infinitesimal generators, conserved vectors, and Euler-Lagrange operators are presented below.

2.1. Infinitesimal Generator. Consider a general system of partial differential equations (PDEs) expressed in n th order involving r independent variables

$$\mathbf{z} = (z^1, z^2, z^3, \dots, z^r)$$

and t dependent variables

$$\mathbf{d} = (d^1, d^2, d^3, \dots, d^t)$$

where $\mu = 1, 2, 3, \dots, t$,

$$F_\mu = (z, d, d_1, d_2, d_3, \dots, d_n). \quad (2.1)$$

Here $d_1, d_2, d_3, \dots, d_n$ denote the n th-order partial derivatives of the dependent variables.

The infinitesimal generator Z , together with its prolongation of order i , is described by

$$Z^{[i]} = Z + \tau_x \frac{\partial}{\partial d_x} + \dots + \tau_{x_1 \dots x_i} \frac{\partial}{\partial d_{x_1 \dots x_i}}, \quad (2.2)$$

where the coefficients τ take the form

$$\begin{aligned}\tau_x^s &= D_x(\eta^s) - d_l^s D_x(\xi^l), \\ &\vdots \\ \tau_{x_1 \dots x_i}^s &= D_{x_i}(\eta_{x_1 \dots x_{i-1}}^s) - d_{kx_1 \dots x_{i-1}}^s D_{x_i}(\xi^k).\end{aligned}$$

2.2. Conserved Vector. A conserved vector is represented as a j -tuple $\mathbf{Y} = (Y^1, Y^2, Y^3, \dots, Y^j)$, which fulfills the condition

$$D_n Y^n = 0, \quad n = 1, 2, 3, \dots, j \quad (2.3)$$

for all admissible solutions of Eq.(2.1).

2.3. Euler-Lagrange Operator. For each component α , the operator

$$E_u = \frac{\delta}{\delta u} = \frac{\partial}{\partial d^\alpha} + \sum_{i \geq 1} (-1)^s D_{x_1} \dots D_{x_i} \tau_{x_1 \dots x_i}^\alpha \frac{\partial}{\partial d_{x_1 \dots x_i}^\alpha}, \quad \alpha = 1, 2, \dots, t, \quad (2.4)$$

is referred to as the Euler-Lagrange operator, or simply the Euler operator.

2.4. Multiplier Function. Let Λ^μ be a multiplier of the form

$$\Lambda^\mu(z, d, d_1, d_2, d_3, \dots, d_n)$$

associated with the conserved vectors of the system (2.1).

It satisfies the relation

$$\Lambda^\mu(F_\mu) = D_n Y_n. \quad (2.5)$$

2.5. Associated Symmetry and Conservation Law. Assume that a generalized symmetry operator $\mathbf{Z}$ corresponds to a conserved quantity $\mathbf{Y}$ of Eq. (2.1)

If $\mathbf{Z}$ and $\mathbf{Y}$ satisfy

$$Z(Y^i) - Y^j D_j(\xi^i) + Y^i D_j(\xi^j) = 0, \quad (2.6)$$

then $\mathbf{Z}$ is said to be related to $\mathbf{Y}$.

The conservation laws obeys the divergence condition

$$D_r Y^r + D_s Y^s = 0,$$

where Y^r and Y^s in terms of variables (t_1, x_1) can be written as

$$Y^r = \frac{Y^{t_1 D_{t_1}(r) + Y^{x_1} D_{x_1}(r)}}{D_{t_1}(r) D_{x_1}(s) - D_{t_1}(s) D_{x_1}(r)}, \quad (2.7)$$

$$Y^s = \frac{Y^{t_1 D_{t_1}(s) + Y^{x_1} D_{x_1}(s)}}{D_{t_1}(r) D_{x_1}(s) - D_{t_1}(s) D_{x_1}(r)}. \quad (2.8)$$

2.6. Theorem 1. Let Y^i denote the conservation laws satisfying $D_i Y^i = 0$.

The mathematical formulation can be expressed as

$$\begin{pmatrix} Y^1 \\ Y^2 \\ \vdots \\ Y^n \end{pmatrix} = J(M^{-1})^T \begin{pmatrix} Y^1 \\ Y^2 \\ \vdots \\ Y^n \end{pmatrix}, \quad J \begin{pmatrix} Y^1 \\ Y^2 \\ \vdots \\ Y^n \end{pmatrix} = M^T \begin{pmatrix} \tilde{Y}^1 \\ \tilde{Y}^2 \\ \vdots \\ \tilde{Y}^n \end{pmatrix}, \quad (2.9)$$

where the matrices M , M^{-1} and the term J are derived from

$$M = \begin{pmatrix} D_1 t_1 & D_1 t_2 & \dots & \dots & \dots & D_1 t_n \\ D_2 t_1 & D_2 t_2 & \dots & \dots & \dots & D_2 t_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ D_n t_1 & D_n t_2 & \dots & \dots & \dots & D_n t_n \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} D_1 t_1 & D_1 t_2 & \dots & \dots & \dots & D_1 t_n \\ D_2 t_1 & D_2 t_2 & \dots & \dots & \dots & D_2 t_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ D_n t_1 & D_n t_2 & \dots & \dots & \dots & D_n t_n \end{pmatrix}, \quad (2.10)$$

and $J = \det(M)$.

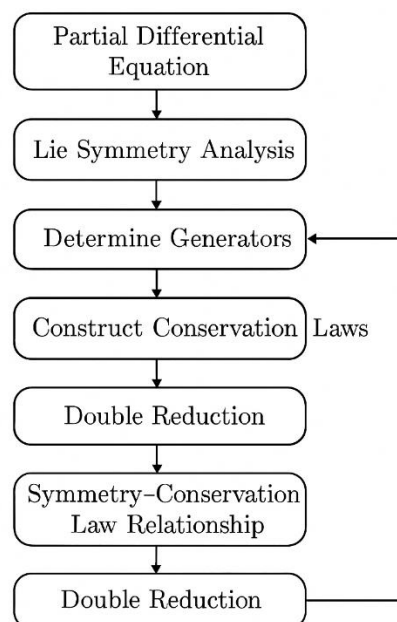


Figure 1. Overall Methodology Framework

3. Lie Symmetries, Conservation Laws and Exact Solutions of NKdV Equation

This section introduces the Lie symmetry analysis, associated conservation laws, and exact analytical solutions of the negative-order Korteweg-de Vries (nKdV) equation using the double-reduction framework. The governing equation is expressed as

$$U_{xxxt} + 4U_x U_{xt} + 2U_{xx} U_t + U_{xx} = 0. \quad (3.1)$$

To determine the Lie symmetries of Eq.(3.1), consider the infinitesimal generator

$$Z = \xi^1(t, x) \frac{\partial}{\partial t} + \xi^2(t, x) \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial U}. \quad (3.2)$$

The symmetry conditions are derived by applying the fourth-order prolongation $Z^{[4]}$ to Eq.(3.1) and by imposing

$$Z^{[4]}[U_{xxxt} + 4U_x U_{xt} + 2U_{xx} U_t + U_{xx}]|_{(11)} = 0, \quad (3.3)$$

yields to an over-determined system for the unknown coefficients ξ^1 , ξ^2 and η .

Solving this system gives the following conditions:

$$\begin{aligned} \xi_u^1 &= 0, & \xi_x^1 &= 0, & \xi_{xx}^1 &= 0, & \xi_{ux}^1 &= 0, \\ \xi_t^2 &= 0, & \xi_u^2 &= 0, & \xi_{ut}^2 &= 0, & \xi_x^2 &= 0, & \eta_{uu} &= 0, & \eta_{uuu} &= 0, \\ \eta_x &= 0, & \eta_{xx} &= 0, & \eta_u &= 0, & \eta_{ut} &= 0, & \eta_{ux} &= 0, & 2\eta_t - \xi_t^1 &= 0. \end{aligned} \quad (3.4)$$

The corresponding Lie-point symmetry generators are

$$\begin{aligned} Z_1 &= \frac{\partial}{\partial t}, & Z_2 &= \frac{\partial}{\partial x}, \\ Z_3 &= \frac{\partial}{\partial U} - 2 \frac{\partial}{\partial t}, & Z_4 &= U \frac{\partial}{\partial U} + t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x}. \end{aligned} \quad (3.5)$$

3.1. Conservation Laws. To construct the conservation laws associated with Eq.(3.1), the multiplier method is adopted. Let $\Lambda = \Lambda(t, x)$ denote an undetermined multiplier function. Its governing condition is established through the relation

$$\frac{\delta}{\delta u} [\Lambda(U_{xxxt} + 4U_x U_{xt} + 2U_{xx} U_t + U_{xx})] = 0, \quad (3.6)$$

where $\frac{\delta}{\delta u}$ denotes the standard Euler operator. By expanding the expression and isolating terms according to the different derivatives of u , the following system of equations is produced:

$$\Lambda_{U_x, U_x} = 0, \Lambda_t = 0, \Lambda_x = 0, \Lambda_U = 0, \Lambda_{U_{xxx}} = 0. \quad (3.7)$$

The resulting system admits the general solution

$$\Lambda = A_1 U_x + A_2, \quad (3.8)$$

where A_1, A_2 represents arbitrary constants. Substituting Eq.(3.8) into the divergence expression gives

$$\Lambda(U_{xxx} + 4U_x U_{xt} + 2U_{xx} U_t + U_{xx}) = D_t Y^t + D_x Y^x, \quad (3.9)$$

for the dependent variable $U(t, x)$.

3.2. Conserved Vectors. From Eq.(3.9) two conserved vectors are obtained and summarized in

Table 1.

TABLE 1. Conserved vectors for Eq.(3.1)

Multipliers	Conserved vector
$A_1 = U_x$	$Y_1^t = \frac{4U_x^3}{3} + \frac{2UU_x U_{xx}}{3} + \frac{U_{xxx} U_x}{2}$ $Y_1^x = \frac{2UU_x U_{tx}}{3} + \frac{2U_x^2 U_t}{3} - \frac{U_{tx} U_{xx}}{2} + \frac{U_x^2}{2} + \frac{U_{txx} U_x}{2}$
$A_2 = 1$	$Y_2^t = 2U_x^2 + UU_{xx} + U_{xxx}$ $Y_2^x = -UU_{tx} + U_x U_t + U_x$

3.3. Reductions and New Exact Solutions. Let $Y_1 = (Y_1^t, Y_1^x)$ and $Y_2 = (Y_2^t, Y_2^x)$. To verify the association between symmetries and the conserved quantities, the relation

$$Z \begin{pmatrix} Y^t \\ Y^x \end{pmatrix} - \begin{pmatrix} D_t \xi^1 & D_x \xi^1 \\ D_t \xi^2 & D_x \xi^2 \end{pmatrix} \begin{pmatrix} Y^t \\ Y^x \end{pmatrix} + (D_t \xi^1 + D_x \xi^2) \begin{pmatrix} Y^t \\ Y^x \end{pmatrix} = 0, \quad (3.10)$$

is employed.

The associated symmetries Z_1 and Z_2 are used for reduction and for constructing exact solutions.

3.3.1. Reduction in Y_1 . Consider the linear combination of symmetries

$$Z = \frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial x}. \quad (3.11)$$

The canonical variables corresponding to Z are obtained from

$$r = -\alpha t + x, \quad s = t, \quad V(r) = U(t, x), \quad (3.12)$$

Using Eq.(2.7) with these transformations yields

$$Y_1^r = \alpha Y_1^t - Y_1^x. \quad (3.13)$$

Substituting the components Y_1^t and Y_1^x from Table 1 gives

$$Y_1^r = \alpha \frac{4}{3} V_r^3 + \alpha \frac{4}{3} V V_r V_{rr} + \alpha V_{rrr} V_r + \alpha \frac{2}{3} V_r V_r^2 - \alpha \frac{1}{2} V_{rr}^2 - \frac{1}{2} V_r^2, \quad (3.14)$$

The reduced conservation form $D_r Y_1^r = 0$ implies $Y_1^r = C_1$, leading to

$$\alpha \frac{4}{3} V_r^3 + \alpha \frac{4}{3} V V_r V_{rr} + \alpha V_{rrr} V_r + \alpha \frac{2}{3} V_r V_r^2 - \alpha \frac{1}{2} V_{rr}^2 - \frac{1}{2} V_r^2 = C_1. \quad (3.15)$$

Solving Eq.(3.15) using Maple yields the following implicit and explicit solutions:

$$V(r) = \pm \sqrt{-2kr} + B, \quad V(r) = B \pm \frac{r}{2\sqrt{\alpha}}. \quad (3.16)$$

Re-expressing the results in the original variables gives

$$U(t, x) = \pm \sqrt{-2k(-\alpha t + x)} + B, \quad U(t, x) = B \pm \frac{(-\alpha t + x)}{2\sqrt{\alpha}}. \quad (3.17)$$

These represent the required exact solutions of the nKdV equation.

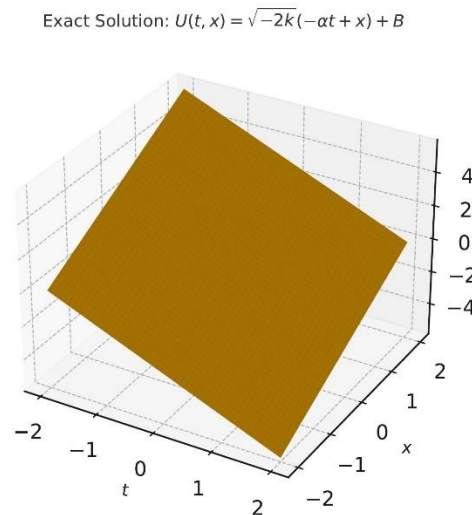


Figure 2. 3D visualization of invariant solution of nKdV equation

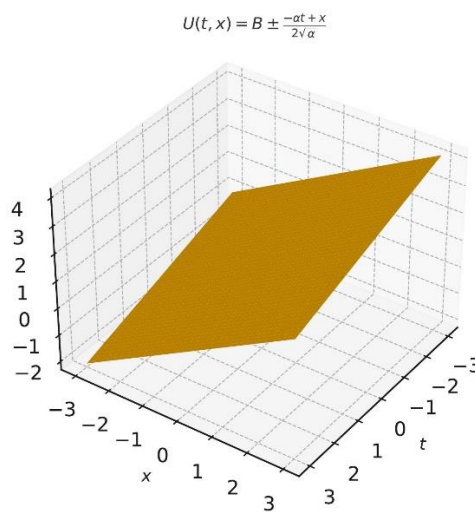


Figure 3. 3D visualization of invariant solution of nKdV equation

3.3.2. Reduction in Y_2 . Similarly, using the conserved vector Y_2 and the combination of symmetries;

$$Z = \frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial x}, \quad (3.18)$$

the corresponding canonical variables are introduced as

$$r = -\alpha t + x, \quad s = t, \quad V(r) = U(t, x), \quad (3.19)$$

Using Eq.(2.7) with these transformations yields

$$Y_2^r = \alpha Y_2^t - Y_2^x, \quad (3.20)$$

Substituting the components Y_1^t and Y_1^x from from Table 1 leads to

$$Y_2^r = 3\alpha V_r^2 + \alpha V_{rrr} - V_r, \quad (3.21)$$

The reduced conservation form $D_r Y_2^r = 0$ gives $Y_2^r = C_2$, or

$$3\alpha V_r^2 + \alpha V_{rrr} - V_r = C_2. \quad (3.22)$$

Integration in Maple produces

$$V(r) = -kr + D, \quad V(r) = D + \frac{r}{3\alpha}. \quad (3.23)$$

Expressed in the original variables, the corresponding solutions are

$$U(t, x) = kat - kx + D, \quad U(t, x) = D + \frac{(-at+x)}{3a}. \quad (3.24)$$

These are additional invariant solutions of the nKdV equation obtained through the double-reduction method.

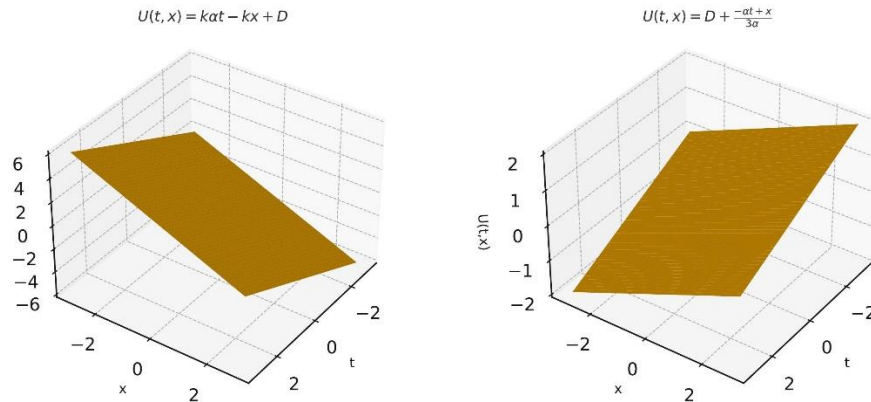


Figure 4. 3D visualization of invariant solutions of nKdV equation

These solutions demonstrate that the proposed symmetry reduction successfully preserves the nonlinear structure of the original equation while reducing its dimensional complexity. Such invariant solutions may provide useful analytical benchmarks for future numerical investigations.

4. Lie Symmetries, Conservation Laws and Exact Solutions of MKdV-KP Equation

This section investigates the Lie-symmetry structure, conservation laws, and invariant solutions of the modified Korteweg-de Vries Kadomtsev-Petviashvili (mKdV-KP) equation. The governing nonlinear evolution equation is expressed as

$$U_{xt} + aU_{xx} + bU^2U_x + cU_{xxxx} + dU_{yy} = 0. \quad (4.1)$$

where a, b, c and d are physical parameters representing dispersion and nonlinearity.

4.1. Lie Symmetry Analysis. To identify the infinitesimal symmetries, the generator is taken in the form

$$Z = \xi^1(t, x, y) \frac{\partial}{\partial t} + \xi^2(t, x, y) \frac{\partial}{\partial x} + \xi^3(t, x, y) \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial U}. \quad (4.2)$$

By applying the fourth prolongation $Z^{[4]}$ to Eq.(4.1) and imposing the condition

$$Z^{[4]}[U_{xt} + aU_{xx} + bU^2U_x + cU_{xxxx} + dU_{yy}]|_{(35)} = 0, \quad (4.3)$$

produces an over-determined system for the coefficients ξ^1, ξ^2 and η . Solving this system yields the following conditions:

$$\begin{aligned} \xi_U^1 &= 0, \quad \xi_x^1 = 0, \quad \xi_U^2 = 0, \quad \xi_U^3 = 0, \quad \xi_x^3 = 0, \\ \xi_y^1 &= 0, \quad \xi_{xU}^1 = 0, \quad \xi_{xU}^2 = 0, \quad \xi_{xxU}^2 = 0, \quad \xi_{xx}^1 = 0, \\ \xi_{xx}^3 &= 0, \quad \eta_{UU} = 0, \quad \eta_{xUU} = 0, \quad \xi_t^1 - 3\xi_x^2 = 0 \\ 6c\xi_{xx}^2 - 4c\eta_{xU} &= 0, \quad 2d\xi_y^3 - 4d\xi_x^2 = 0, \quad 2d\xi_y^2 + \xi_t^3 = 0, \end{aligned} \quad (4.4)$$

$$\begin{aligned} d\xi_{yy}^3 - 2d\eta_{yU} &= 0, \quad d\xi_{yy}^1 - \eta_{xU} = 0, \quad \eta_{xt} + a\eta_{xx} + bU^2\eta_x \\ &+ c\eta_{xxxx} + d\eta_{yy} = 0, \quad \xi_t^2 - 2a\xi_x^2 + 4c\xi_{xxx}^2 - 6c\eta_{xxU} = 0, \\ \xi_{xt}^2 + a\xi_{xx}^2 + d\xi_{yy}^2 + c\xi_{xxx}^2 - \eta_{tU} - 2a\eta_{xU} - 4c\eta_{xxxU} - 2bU\eta - 3bU^2\xi_x^2 &= 0. \end{aligned}$$

The corresponding Lie-point symmetry generators are

$$\begin{aligned} Z_1 &= \frac{\partial}{\partial t}, \quad Z_2 = \frac{\partial}{\partial x}, \\ Z_3 &= \frac{\partial}{\partial y}, \quad Z_4 = y \frac{\partial}{\partial x} - 2dt \frac{\partial}{\partial y}, \end{aligned} \quad (4.5)$$

$$Z_5 = -\frac{3U}{2} \frac{\partial}{\partial U} + 3t \frac{\partial}{\partial t} + (x + 2at) \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}.$$

These generators describe temporal translation, spatial translations in x and y Galilean-type transformations, and scaling symmetries, respectively.

4.2. Conservation Laws via the Multiplier Method. To construct the conservation laws, the multiplier function $\Lambda = \Lambda(t, x, y)$ is introduced. The determining equation is obtained from

$$\frac{\delta}{\delta u} [\Lambda (U_{xt} + aU_{xx} + bU^2U_x + cU_{xxx} + dU_{yy})] = 0, \quad (4.6)$$

here $\frac{\delta}{\delta u}$ is the standard Euler operator.

Separation with respect to the independent derivatives of u gives

$$\Lambda_{y,y} = 0, \Lambda_x = 0, \Lambda_U = 0, \Lambda_{Uy} = 0, \quad (4.7)$$

The general solution of this system takes the form

$$\Lambda = y + 1. \quad (4.8)$$

Substituting Eq.(4.8) into the divergence expression yields

$$\Lambda (U_{xt} + aU_{xx} + bU^2U_x + cU_{xxx} + dU_{yy}) = D_t Y^t + D_x Y^x + D_y Y^y, \quad (4.9)$$

for all admissible functions $U(t, x, y)$.

4.3. Conserved Vectors. Two independent conserved vectors are obtained and summarized below.

TABLE 2. Conserved vectors for Eq.(4.1)

Multipliers	Conserved vector	
$\Lambda_1 = y$	$Y_1^t = U_x y$ $Y_1^x = \frac{y}{3} (U^3 b + 3U_x a + 3U_{xxx} c),$	$Y_1^y = -Ud + U_y yd$
$\Lambda_2 = 1$	$Y_2^t = U_x$ $Y_2^x = \frac{U^3 b}{3} + U_x a + U_{xxx} c,$	$Y_2^y = U_y d$

4.4. Reduction and Invariant Solutions. To establish the correspondence between the symmetries and the conservation laws, the relation

$$Z \begin{pmatrix} Y^t \\ Y^x \\ Y^y \end{pmatrix} - \begin{pmatrix} D_t \xi^1 & D_x \xi^1 & D_y \xi^1 \\ D_t \xi^2 & D_x \xi^2 & D_y \xi^2 \\ D_t \xi^3 & D_x \xi^3 & D_y \xi^3 \end{pmatrix} \begin{pmatrix} Y^t \\ Y^x \\ Y^y \end{pmatrix} + (D_t \xi^1 + D_x \xi^2 + D_y \xi^3) \begin{pmatrix} Y^t \\ Y^x \\ Y^y \end{pmatrix} = 0, \quad (4.10)$$

Only the associated symmetries listed below are employed to obtain the exact solutions of the considered PDE.

4.4.1. Reduction in Y_2 . The combined symmetries used for reduction are Z_1, Z_2, Z_3

$$Z = \frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y}. \quad (4.11)$$

The corresponding canonical variables are defined as

$$r = y - \beta t, \quad s = x - \alpha t, \quad m = t, \quad V(r, s) = U(t, x, y). \quad (4.12)$$

Substituting these transformations and using Eq.(2.7) provides the reduced conserved form

$$D_r Y_2^r + D_s Y_2^s = 0. \quad (4.13)$$

with

$$Y_2^r = -\beta V_s + V_r d, \\ Y_2^s = -\alpha V_s + \frac{V^3 b}{3} + aV_s + cV_{sss}, \quad (4.14)$$

Applying the double-reduction analysis decreases both the order and the dimensionality of the PDE. Introducing an additional symmetry combination

$$Z = \frac{\partial}{\partial r} + \gamma \frac{\partial}{\partial s}, \quad (4.15)$$

and the canonical coordinates are

$$n = r, \quad q = s - \gamma r, \quad W(q) = V(r, s). \quad (4.16)$$

transforms Eq.(4.13) into

$$D_q Y_2^q = 0, \quad (4.17)$$

where

$$Y_2^q = -\gamma^2 W_q d - \gamma \beta W_q - \frac{W^3 b}{3} - a W_q - c W_{qqq} + \alpha W_q, \quad (4.18)$$

Integrating yields

$$-\gamma^2 W_q d - \gamma \beta W_q - \frac{W^3 b}{3} - a W_q - c W_{qqq} + \alpha W_q = H. \quad (4.19)$$

where H is a constant of integration.

Equation (4.19) is solved using Maple, leading to the following implicit form:

$$W(q) = \frac{1}{\vartheta} [-E\sqrt{c}\cos(\phi q) + F\sqrt{c}\sin(\phi q) - kq + G], \quad (4.20)$$

where $\phi = \frac{\sqrt{\gamma^2 d + \gamma \beta + a - \alpha}}{\sqrt{c}}$ and $\vartheta = \sqrt{\gamma^2 d + \gamma \beta + a - \alpha}$.

Reverting to the original variables gives the invariant solution

$$U(t, x, y) = \frac{1}{\vartheta} [-E\sqrt{c}\cos\phi(x - at - \gamma y + \gamma \beta t) + \quad (4.21)$$

$$F\sqrt{c}\sin\phi(x - at - \gamma y + \gamma \beta t) - k(x - at - \gamma y + \gamma \beta t) + G],$$

These solutions demonstrate the effectiveness of the double-reduction technique in constructing new analytical forms of multidimensional nonlinear evolution equations.

$$U(t, x, y) = \frac{1}{\vartheta} [-E\sqrt{c}\cos(\phi\Psi) + F\sqrt{c}\sin(\phi\Psi) - k\Psi + G] \quad \dots \Psi = x - at - \gamma y + \gamma \beta t$$

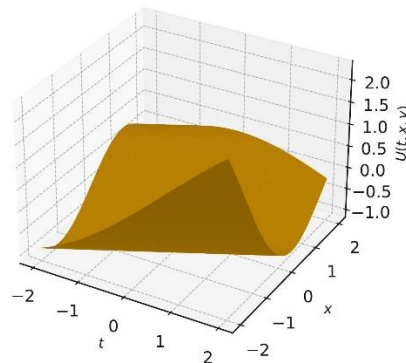


Figure 5. 3D visualization of invariant solution of MkdV-KP equation

These solutions demonstrate that the proposed symmetry reduction successfully preserves the nonlinear structure of the original equation while reducing its dimensional complexity. Such invariant solutions may provide useful analytical benchmarks for future numerical investigations.

5. Lie Symmetries, Conservation Laws and Exact Solutions of Generalized KdV Like Model Equation

This section outlines the Lie-symmetry structure, the associated conservation laws, and the invariant solutions corresponding to a generalized Korteweg–de Vries-type model equation. The governing multidimensional nonlinear evolution equation is

$$U_t + aUU_x + aU_{xxx} + b(hUU_y + U_{yxx}) + cgUU_z + U_{zzx} = 0. \quad (5.1)$$

5.1. Lie Symmetry Analysis. To determine the infinitesimal symmetries of Eq.(5.1), consider the generator

$$Z = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \xi^4 \frac{\partial}{\partial z} + \eta \frac{\partial}{\partial U}. \quad (5.2)$$

Applying the third prolongation $Z^{[3]}$ to Eq.(5.1) and requiring invariance under Z yields

$$Z^{[3]}[U_t + aUU_x + aU_{xxx} + b(hUU_y + U_{yxx}) + cgUU_z + U_{zzx}]|_{(5.1)} = 0, \quad (5.3)$$

which produces an over-determined system for the unknown functions $\xi_1, \xi_2, \xi_3, \xi_4, \eta$.

After simplification, the system of determining equations is obtained as

$$\begin{aligned} \xi_U^1 &= 0, \quad \xi_x^1 = 0, \quad \xi_{xx}^1 = 0, \quad \xi_{xU}^1 = 0, \quad \xi_U^2 = 0, \\ \xi_x^3 &= 0, \quad \xi_{xx}^3 = 0, \quad \xi_{xU}^3 = 0, \quad \xi_U^3 = 0, \quad \xi_x^4 = 0, \\ \xi_{xx}^4 &= 0, \quad \xi_{xU}^4 = 0, \quad \xi_U^4 = 0, \quad \eta_{UU} = 0, \quad \eta_{xUU} = 0, \\ ca\xi_z^1 + ba\xi_y^1 &= 0, \quad ba\xi_{xx}^2 - 2ab\eta_{xU} = 0, \quad c\xi_{xx}^2 - 2ca\eta_{xU} = 0, \\ -b^2\xi_y^2 + ca\xi_z^3 + ba\xi_y^3 - ba\xi_x^2 - cb\xi_z^2 &= 0, \\ -c^2\xi_z^2 + ba\xi_y^4 + ca\xi_z^4 - ca\xi_x^2 - bc\xi_y^2 &= 0, \\ -3a^2\eta_{xU} + 3a^2\xi_{xx}^2 - ca\eta_{zU} - ba\eta_{yU} + 2ba\xi_{yx}^2 + 2ca\xi_{zx}^2 &= 0, \\ cgUa\eta_z + 6a^2U\eta_x + ba\eta_{yxx} + ca\eta_{zxx} + a\eta_t + a^2\eta_{xx} + bhUa\eta_y &= 0, \\ bhUa\xi_y^1 + cgUa\xi_z^1 - b\xi_y^2 - c\xi_z^2 + a\xi_t^1 - 3a\xi_x^2 &= 0, \\ a\xi_t^3 - b^2hU\xi_y^2 - bha\eta - ba\eta_{xxU} - cbhU\xi_z^2 + abhU\xi_y^3 + acgU\xi_z^3 - 3bhUa\xi_z^2 &= 0, \\ a\xi_t^4 - c^2gU\xi_z^2 - cga\eta - 3cgaU\xi_x^2 - bcgU\xi_y^2 + bhUa\xi_y^4 + acgU\xi_z^4 - ca\eta_{xxU} &= 0, \\ -3a^2\eta_{xxU} - 6a^2\eta + a^2\xi_{xxx}^2 + a\xi_t^2 - 6abU\xi_y^2 - 6cUa\xi_z^2 - 2ca\eta_{zxU} + ba\xi_{yxx}^2 + ca\xi_{zxx}^2 \\ -2ba\eta_{yxU} - 12a^2U\xi_x^2 + bhUa\xi_y^2 + cgUa\xi_z^2 &= 0. \end{aligned} \quad (5.4)$$

Solving Eq.(5.4) produces the following Lie-point symmetry generators:

$$\begin{aligned} Z_1 &= \frac{\partial}{\partial t}, \quad Z_2 = \frac{\partial}{\partial x}, \quad Z_3 = b \partial \partial y + c \frac{\partial}{\partial z}, \\ Z_4 &= (2Ucbg - 2Ucbh) \frac{\partial}{\partial U} + (2tcbh - 2ctbg) \frac{\partial}{\partial t} \\ &+ (6zba + gcya + xchb - zbah - 6cya - xgcb) \frac{\partial}{\partial x}, \\ Z_5 &= \frac{\partial}{\partial U} + 6at \frac{\partial}{\partial x} + bht \frac{\partial}{\partial y} + tgc \partial \partial z. \end{aligned} \quad (5.5)$$

These operators correspond to temporal and spatial translations, scaling, and Galilean-type transformations, reflecting the multidimensional and coupled structure of Eq.(5.1).

5.2. Conservation Laws via the Multiplier Method. Let the multiplier function be $\Lambda = \Lambda(t, x, y, z)$. The determining equation for Λ follows from

$$\frac{\partial}{\partial u} [\Lambda(U_t + aUU_x + aU_{xxx} + b(hUU_y + U_{yxx}) + cgUU_z + U_{zxx})] = 0, \quad (5.6)$$

Upon expanding the determining equation and isolating terms according to the independent derivatives of u the resulting system becomes

$$\begin{aligned} \Lambda_{y,y,y} &= \frac{1}{6b^3h^2 - b^3h^3} (-1296\Lambda_x a^3U - 216a^2\Lambda_y bhU - 216a^2\Lambda_t + 3bc^2\Lambda_{y,z,z}g^2h \\ &- 216a^2\Lambda_z cgU - 12\Lambda_{y,y,z}cgb^2h + 3\Lambda_{y,y,z}cgb^2h^2 - 12bc^2\Lambda_{y,z,z}gh - 6c^3g^2\Lambda_{z,z,z} \\ &- 6b^2c\Lambda_{y,y,z}h^2 - 6bc^2\Lambda_{y,z,z}g^2 + c^3g^3\Lambda_{z,z,z}), \\ \Lambda_{x,x} &= \frac{1}{36a^2} (c^2g^2\Lambda_{z,z} + 2cg\Lambda_{y,z}bh + b^2h^2\Lambda_{y,y}), \quad \Lambda_{x,y} = \frac{-1}{6a} (\Lambda_{y,z}cg + \Lambda_{y,y}bh), \end{aligned} \quad (5.7)$$

$$\begin{aligned} \Lambda_{x,z} &= \frac{1}{6a} (-\Lambda_{z,z}cg - \Lambda_{y,z}bh), \quad \Lambda_{t,x} = \frac{1}{6a} (-bh\Lambda_{t,y} - cg\Lambda_{t,z}), \quad \Lambda_{U,x} = 0, \\ \Lambda_{U,y} &= 0, \quad \Lambda_{U,z} = 0, \quad \Lambda_{U,U} = 0, \quad \Lambda_{Uxxz} = 0, \quad \Lambda_{U,t} = -6\Lambda_x a - \Lambda_z cg - \Lambda_y bh. \end{aligned}$$

The resulting expression for the multiplier is obtained as

$$\Lambda = \frac{6Uta-x}{6a} + U, \quad (5.8)$$

Substituting Eq.(5.8) into the divergence relation gives

$$\begin{aligned} \Lambda(U_t + aUU_x + aU_{xxx} + b(hUU_y + U_{yxx}) \\ + cgUU_z + U_{zxx}) &= D_t Y^t + D_x Y^x + D_y Y^y + D_z Y^z, \end{aligned} \quad (5.9)$$

for all admissible functions $U(t, x, y, z)$.

5.3. Conserved Vectors. The associated conserved vectors derived from Eq.(5.9) are summarized below.

TABLE 3. Conserved vectors For Eq.(5.1)

Multipliers	Conserved vectors
$\Lambda_1 = \frac{6Uta - x}{6a}$	$Y_1^t = \frac{U(6Uta - x)}{6a}$ $Y_1^x = \frac{1}{6a} (12U^3ta^2 + 6ta^2UU_{xx} - 3ta^2U_x^2 - 3U_ytbaU_x - 3tcaU_zU_x + 3U_{xy}tUba + 3U_{xz}tUca + aU_x - U_{xz}xc + U_zc + U_yb - U_{xy}xb - l$ $Y_1^y = \frac{Ub}{12a} (4U^2tha - Uxh + 6taU_{xx})$ $Y_1^z = \frac{Uc}{12a} (4U^2tga - Uxg + 6taU_{xx})$
$\Lambda_2 = U$	$Y_2^t = \frac{U^2}{2}$ $Y_2^x = 2U^3a + aUU_{xx} - \frac{U_x^2a}{2} - \frac{U_ybU_x}{2} - \frac{U_zcU_x}{2} + \frac{U_{xy}Ub}{2} + \frac{U_{xz}l}{2}$ $Y_2^y = \frac{U^3bh}{3} + \frac{UbU_{xx}}{2}$ $Y_2^z = \frac{U^3cg}{3} + \frac{UU_{xx}c}{2}$

5.4. Reduction and Exact Solutions. To establish the relationship between the symmetry structure and the associated conservation laws, the following identity is employed

$$Z \begin{pmatrix} Y^t \\ Y^x \\ Y^y \\ Y^z \end{pmatrix} - \begin{pmatrix} D_t \xi^1 & D_x \xi^1 & D_y \xi^1 & D_z \xi^1 \\ D_t \xi^2 & D_x \xi^2 & D_y \xi^2 & D_z \xi^2 \\ D_t \xi^3 & D_x \xi^3 & D_y \xi^3 & D_z \xi^3 \\ D_t \xi^4 & D_x \xi^4 & D_y \xi^4 & D_z \xi^4 \end{pmatrix} \begin{pmatrix} Y^t \\ Y^x \\ Y^y \\ Y^z \end{pmatrix} + (D_t \xi^1 + D_x \xi^2 + D_y \xi^3 + D_z \xi^4) \begin{pmatrix} Y^t \\ Y^x \\ Y^y \\ Y^z \end{pmatrix} = 0, \quad (5.10)$$

Only the following associated symmetries are used for the exact solutions of considered PDE

5.4.1. Reduction in Y_2 . To derive invariant solutions, consider a combination of the symmetries

$$Z = \alpha \frac{\partial}{\partial t} + \beta \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z}. \quad (5.11)$$

The canonical variables corresponding to Z are obtained as

$$r = \frac{-bt+y\alpha}{\alpha}, \quad s = \frac{-ct+z\alpha}{\alpha}, \quad m = \frac{-\beta t+x\alpha}{\alpha}, \quad n = \frac{t}{\alpha}, \quad V(r, s, m) = U(t, x, y, z). \quad (5.12)$$

Substitution into Eq.(5.9) and use of Eq.(2.9) yields the reduced conservation form

$$D_k Y_2^k = 0, \quad (5.13)$$

with the corresponding reduced equation

$$\frac{1}{2\alpha} (\zeta \sigma b - \beta - \sigma b - c) Y^2 + \frac{1}{3} (\zeta \sigma b h + 6\alpha - \sigma b h + c g) Y^3 + \frac{\sigma b}{2} (1 - \zeta^2) Y Y_k + \left(a \zeta^2 - \frac{c \zeta}{2} \right) Y Y_{kk} + \frac{1}{2} (2a \sigma \zeta + \sigma b \zeta^2 + c \zeta - a \zeta^2 \sigma^2 - a \gamma^2 - b \zeta \gamma) Y_k^2 = K. \quad (5.14)$$

where K denotes an arbitrary constant of integration.

5.5. Analytical Solution.

Solving Eq.(5.14) using Maple provides the implicit analytical form

$$Y(k) = \frac{1}{4\alpha^2} \left[\beta + \frac{1}{4\alpha^2\alpha^2\gamma^2} \beta \tan^2 \left(\sqrt{\alpha\alpha\beta}(-k + L) \right)^2 \right]. \quad (5.15)$$

Expressing the result in the original variables yields the corresponding invariant solution

$$U(t, x, y, z) = \frac{1}{4\alpha^2} \left[\beta + \frac{1}{4\alpha^2\alpha^2\gamma^2} \beta \tan^2 \left(\sqrt{\alpha\alpha\beta}(-\psi + L) \right)^2 \right]. \quad (5.16)$$

where $\psi = -\zeta \left[\frac{-\sigma(y\alpha - bt) + (\alpha x - \beta t)}{\alpha} \right] - \gamma \left[\frac{\alpha(y+z) - t(b+c)}{\alpha} \right]$

This solution defines a new class of invariant waveforms for the generalized KdV-like model equation, describing the propagation of nonlinear dispersive waves in multidimensional media.

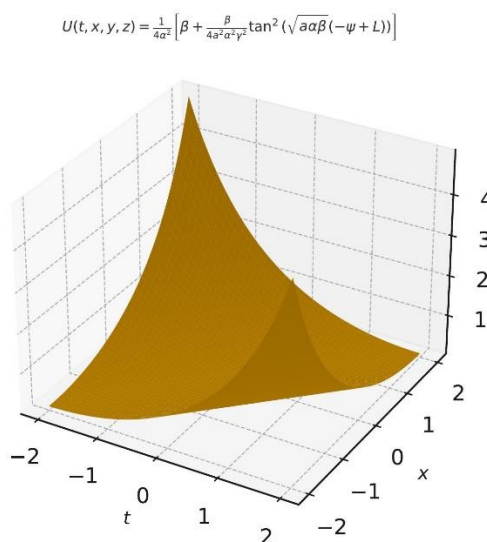


Figure 6. 3D visualization of invariant solution of generalized KdV like model equation

These solutions demonstrate that the proposed symmetry reduction successfully preserves the nonlinear structure of the original equation while reducing its dimensional complexity. Such invariant solutions may provide useful analytical benchmarks for future numerical investigations.

6. Conclusion

This study presented a unified symmetry-based framework for the analytical investigation of three multidimensional KdV-type nonlinear evolution equations in (1+1), (1+2), and (1+3) dimensions. Lie point symmetries were systematically determined for each model, while the direct multiplier method was employed to derive the corresponding conservation laws without requiring a variational formulation. By integrating these results with the double-reduction approach, several invariant reductions together with implicit and explicit exact solutions were successfully obtained.

The proposed framework provides a consistent analytical procedure for studying

multidimensional nonlinear evolution equations through the combined application of symmetry analysis, conservation laws, and double reduction. The obtained results demonstrate the effectiveness of this methodology in reducing the complexity of nonlinear models while preserving their fundamental mathematical structure. To the best of the author's knowledge, the unified treatment presented in this work has received limited attention in the existing literature and offers a useful basis for the analytical investigation of related KdV-type equations.

The methodology developed in this study may be extended to other classes of nonlinear evolution equations, including higher-dimensional, coupled, and fractional-order models, providing further opportunities for the construction of exact analytical solutions and conservation laws

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