

ADVANCED ANALYTICAL AND COMPUTATIONAL FRAMEWORK FOR
FRACTIONAL INTEGRAL INEQUALITIES USING MACHINE LEARNING-
ENHANCED NUMERICAL METHODS FOR NONLINEAR DYNAMICAL
SYSTEMS AND OPTIMIZATION PROBLEMS

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Abstract

Fractional integral inequalities provide essential tools for error estimation, stability analysis, and optimization in memory-dependent systems, but classical Hermite–Hadamard-type bounds can be conservative and costly to evaluate numerically. This study develops an integrated analytical-computational framework combining fractional inequality theory with machine learning-enhanced numerical methods to improve bound tightness, approximation accuracy, and computational efficiency. The framework incorporates Riemann–Liouville, Caputo, Caputo–Fabrizio, and Atangana–Baleanu operator families and supports convex, s-convex, and h-convex functions. Its architecture combines fractional operator embedding, neural residual calibration, and an inequality-constrained loss function to preserve analytically established bound directions while learning tighter estimates. Synthetic function families were evaluated across fractional orders of 0.1–0.9 and convexity parameters of 0.1–1.0, with classical quadrature, high-precision numerical references, and component-level ablations used for benchmarking.

The proposed framework achieved an average bound-gap reduction of 55.0%, with a peak of 65% and mean tightening of 60.7% in the low-order, low-convexity-parameter region. At $s = 0.2$ and 0.8 , bound gaps decreased by 44.2% and 32.7%, respectively. Empirical convergence order increased from

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approximately 2.00 to 2.16, while maximum approximation error decreased by 55% at 128 discretization nodes and by more than 60% at 1,024 nodes. Computational speed-up reached 9.0-fold at one million evaluation points. In a representative fractional-order nonlinear oscillator, the learned controller changed largest Lyapunov exponents from positive values of 0.04–0.21 to negative values of -0.10 to -0.05 across orders of 0.5–1.0. The inequality-guided optimizer reached the target tolerance in approximately 102 iterations, whereas Adam, particle swarm optimization, and gradient descent did not reach it within 200 iterations. Ablation analysis identified the inequality-constrained loss and operator embedding as the largest contributors, with their removal reducing average tightening by 16.5 and 13.8 percentage points, respectively. These findings demonstrate promising gains in fractional numerical analysis, chaos suppression, and constrained optimization, subject to further validation beyond synthetic functions and the representative oscillator. Future work should examine broader function families, additional dynamical systems, and application-specific generalization under realistic computational conditions.

INTRODUCTION

Fractional calculus extends the classical operations of differentiation and integration to arbitrary, non-integer order, and in doing so furnishes a mathematical language uniquely suited to systems whose present behavior depends on an extended history of past states. This memory-encoding property, largely absent from integer-order calculus, has made fractional operators indispensable across viscoelastic modelling, anomalous diffusion, control engineering, epidemiological forecasting, and financial risk analysis, and the breadth of this applicability continues to motivate sustained theoretical investigation into the analytical structures that govern fractional operators [1]. Among these structures, fractional integral inequalities, and the Hermite-Hadamard inequality in particular, occupy a position of special importance because they translate the abstract convexity properties of a function into concrete, computable bounds on its fractional integral average. These bounds underpin error estimates in numerical quadrature, stability certificates in control design, and feasibility guarantees in constrained optimization, making the sharpness and computability of

fractional integral inequalities a matter of direct practical consequence rather than purely theoretical interest [2].

The classical apparatus for deriving fractional integral inequalities, built upon convexity assumptions and refined through successive generalizations across an expanding family of fractional operators, has matured considerably over the past decade, yet it remains constrained in two interlocking respects. First, the analytical bounds produced by classical techniques are typically derived for worst-case configurations within a convexity class and consequently leave a non-trivial gap between the proven upper and lower estimates, a gap that widens precisely in the strongly convex, low fractional-order regime where many physically motivated models reside. Second, the numerical schemes conventionally used to evaluate fractional operators and to verify these inequalities computationally rely on fixed-order quadrature and discretization rules whose convergence behavior, while well understood, does not adapt to the local structure of the function being integrated, leaving accuracy and computational cost in a fixed and often

unfavorable relationship. Recent advances in machine learning, and in physics-informed neural architectures specifically, have demonstrated that learned numerical surrogates can approximate fractional operator evaluations with adaptive, data-driven accuracy that does not share the rigid convergence profile of classical schemes [3], while parallel developments in neural-network-based stabilization of nonlinear fractional-order dynamical systems have shown that learned control policies can suppress chaotic behavior that classical linear controllers struggle to address [4]. A further parallel literature on machine learning-augmented and metaheuristic optimization algorithms has established that gradient information can be productively combined with adaptive, learned search strategies to accelerate convergence on non-convex objectives [5].

What remains comparatively underexplored, and what motivates the present study, is the deliberate integration of these three developments, sharper fractional inequality theory, machine learning-enhanced numerical evaluation, and learned stabilization or optimization strategies, within a single coherent analytical-computational framework rather than as three separate strands of research pursued in isolation. A framework that allows a neural calibration mechanism to learn corrections to classical Hermite-Hadamard bounds, while simultaneously serving as a fast surrogate for fractional operator evaluation and as a foundation for inequality-constrained stabilization and optimization routines, would extend the practical reach of fractional inequality theory considerably beyond what purely analytical or purely numerical treatments can achieve individually. This motivates the central research question addressed in this paper: can a machine learning-enhanced numerical framework simultaneously tighten classical fractional integral inequality bounds, accelerate the computational evaluation of fractional operators, and deliver measurable benefits when applied to nonlinear dynamical system stabilization and constrained optimization, and if so, which architectural

components of such a framework contribute most to these gains.

The breadth of established and emerging applications of fractional calculus, spanning ecological and epidemiological modeling through control-theoretic and game-theoretic formulations, provides the applied backdrop against which this question is posed, and underscores those improvements to the underlying inequality and numerical machinery carry consequences across this entire applied landscape rather than being confined to a narrow theoretical niche [6]. The present study responds to this motivation by developing a hybrid framework in which a neural calibration layer, trained across a structured grid of fractional orders and convexity parameters, learns to tighten classical Hermite-Hadamard-type bounds for several major fractional operator families, while the same underlying architecture is repurposed as a numerical surrogate solver and as the computational backbone for a fractional-order chaos stabilization controller and an inequality-constrained optimization routine.

The remainder of this paper is structured to develop and substantiate this contribution systematically. Section 2 reviews the literature on classical and generalized fractional integral inequalities, machine learning-enhanced numerical methods for fractional operator equations, and the intersection of fractional calculus with nonlinear dynamics and optimization, identifying the specific gap that motivates the proposed framework. Section 3 details the research methodology, describing the six-stage analytical-computational pipeline that links operator formulation, theoretical bound derivation, machine learning-enhanced numerical scheme design, algorithm implementation, validation, and downstream application. Section 4 presents the results of this evaluation across convergence behavior, bound tightening, computational efficiency, dynamical system stabilization, and optimization performance, together with an ablation analysis isolating the contribution of individual framework

components. Section 5 concludes with the complete list of references that situate this contribution within the broader literature.

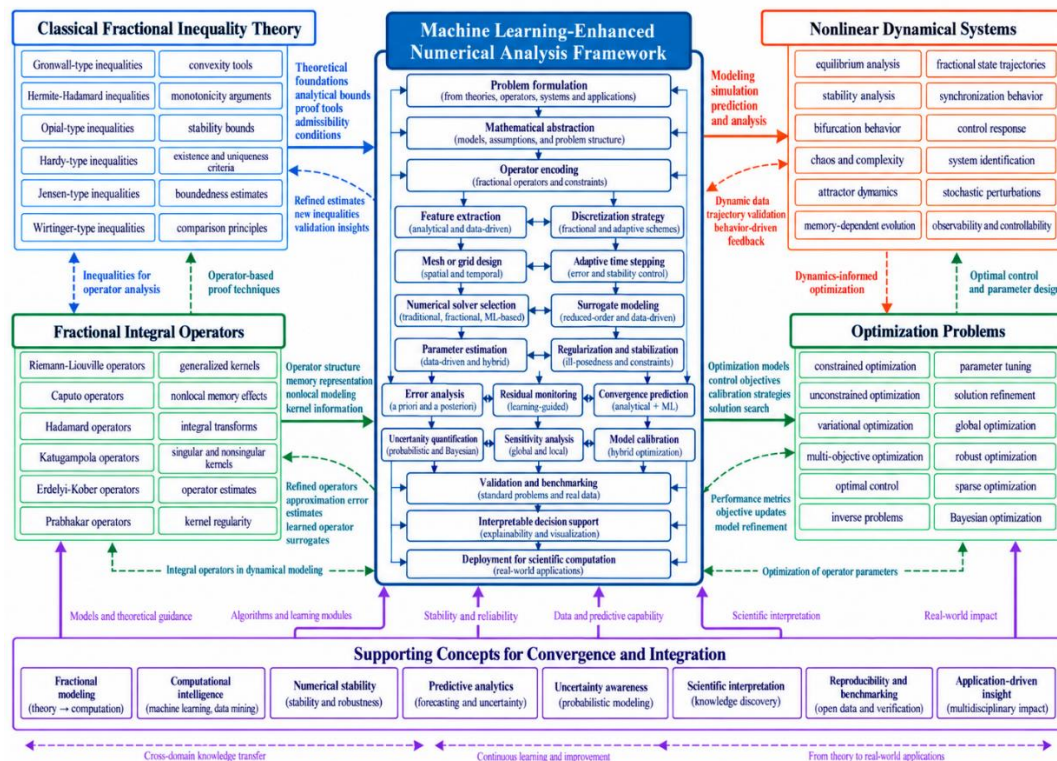


Fig. 1. Integrated conceptual framework illustrating the convergence of fractional inequality theory, fractional integral operators, nonlinear dynamical systems, and optimization through machine learning-enhanced numerical analysis, with interconnected computational processes, feedback mechanisms, and supporting research concepts.

Fig. 1 presents the integrated conceptual framework underlying the present study, showing how four interconnected research domains contribute to a unified machine learning-enhanced numerical analysis approach. Classical fractional inequality theory provides analytical bounds, convexity principles, stability criteria, and existence and uniqueness conditions, while fractional integral operators supply the mathematical representations required to model nonlocal behavior and memory-dependent processes. These theoretical foundations are connected to nonlinear dynamical systems and optimization problems through a central computational framework that combines

mathematical abstraction, operator encoding, feature extraction, adaptive discretization, numerical solver selection, surrogate modeling, parameter estimation, and regularization. Subsequent stages incorporate error analysis, residual monitoring, convergence prediction, uncertainty quantification, sensitivity analysis, and model calibration to strengthen numerical reliability and computational efficiency. The framework further includes validation and benchmarking, interpretable decision support, and deployment for scientific computation. Solid arrows represent the principal flow of theoretical information and computational outputs, whereas dashed connections indicate feedback,

refinement, and knowledge transfer among the research domains. The lower supporting band highlights the complementary roles of fractional modeling, computational intelligence, numerical stability, predictive analytics, uncertainty awareness, scientific interpretation, and reproducibility. Collectively, these components establish a structured pathway from mathematical theory to data-driven approximation and practical applications, providing the conceptual basis for the quantitative analyses presented in Section 4.

Literature Review

Research on fractional integral inequalities and their computational treatment has developed along three increasingly interconnected tracks: the classical and generalized analytical study of convexity-based inequalities, the design of machine learning-enhanced numerical methods for evaluating fractional operator equations, and the application of fractional-order reasoning to nonlinear dynamical systems and optimization. Each track is examined in turn before the discussion identifies the specific methodological gap that the present framework addresses.

Classical and Generalized Fractional Integral Inequalities

The analytical study of fractional integral inequalities has its modern origin in the reformulation of the classical Hermite-Hadamard inequality through fractional integral operators, and the resulting literature has expanded along two complementary directions: the introduction of new fractional operators and the generalization to broader convexity classes. Hezenci, Budak, Kara, and Sarikaya established generalized fractional midpoint-type inequalities for co-ordinated convex functions, deriving identities that connect midpoint estimates to the underlying fractional integral structure for functions of two variables [2]. The same research group subsequently extended this analysis to trapezoid-type inequalities for conformable fractional integrals, demonstrating that the conformable operator, despite its comparatively simple

algebraic form, supports inequality structures closely analogous to those obtained for Riemann-Liouville and Caputo operators [7]. Kara, Budak, Etemad, Rezapour, Ahmad, and Kaabar broadened this line of inquiry further by considering midpoint-type and trapezoidal-type inequalities for twice differentiable functions under conformable operators, showing that second-order differentiability assumptions permit sharper error characterizations than first-order treatments alone [8]. Akdemir, Aslan, Dokuyucu, and Çelik pursued a parallel generalization by examining exponentially convex functions on the coordinates, deriving novel estimations via the Riemann-Liouville fractional operator that extend classical convexity-based bounds to a richer function class capable of modeling asymmetric growth behavior [9].

A distinct but related body of work has focused on specific inequality types, namely Ostrowski, Milne, Maclaurin, Newton, and Bullen-type inequalities, each of which imposes a different sampling or weighting structure on the function under study and consequently yields a different bound profile. Vivas-Cortez, Samraiz, Ullah, Iqbal, and Mukhtar proposed a modified class of Ostrowski-type inequalities together with refined error bounds for Hermite-Hadamard inequalities, illustrating that error bound refinement and inequality generalization are mutually reinforcing research directions [10]. Lakhdari, Budak, Awan, and Meftah extended Milne-type inequalities to the Katugampola fractional integral, a unifying operator that subsumes Riemann-Liouville and Hadamard fractional integrals as special cases, thereby demonstrating that inequality results proven for a sufficiently general operator automatically specialize to a wide range of previously studied operators [12]. The multiplicative calculus variant of this literature, in which integration and differentiation are defined with respect to multiplicative rather than additive structure, has also seen substantial recent activity: Merad, Meftah, Moumen, and Bouye derived fractional Maclaurin-type inequalities for multiplicatively convex functions [13], Boulares,

Meftah, Moumen, Shafqat, Saber, Alraqad, and Ali established fractional multiplicative Bullen-type inequalities for multiplicative differentiable functions [14], and Lakhdari, Benchettah, and Meftah further derived fractional multiplicative Newton-type inequalities for multiplicatively s -convex positive functions with explicit application examples [15]. Samraiz, Ashraf, Naheed, and Vivas-Cortez extended this multiplicative framework to tempered fractional Newton-type inequalities for twice differentiable functions exhibiting multiplicatively inverse cosine convexity, illustrating the continuing diversification of convexity classes under active investigation [11]. Bayraktar, Nápoles, and Rabossi contributed new estimates for integral inequalities with accompanying applications that connect abstract bound derivation to concrete numerical illustration [16], while Mateen, Zhang, Ali, and Fečkan generalized several integral inequalities within multiplicative calculus and paired their theoretical results with explicit computational analysis, signaling a gradual shift toward treating computational verification as an integral part of inequality research rather than a peripheral afterthought [17]. Ali, Zhang, and Fečkan's earlier error-bound analysis for Milne's formula in fractional calculus remains a frequently invoked methodological reference within this body of work, particularly for its treatment of higher-order error terms [18].

Machine Learning-Enhanced Numerical Methods for Fractional Operators

The second major track concerns machine learning-enhanced numerical methods for fractional operator equations, a literature that has grown rapidly alongside the broader maturation of physics-informed neural network architectures. Wang, Zhang, and Jiang developed a physics-informed neural network algorithm for forward and inverse problems involving variable-order space-fractional advection-diffusion equations, demonstrating that a single network architecture can recover both the solution field and the underlying variable fractional order from sparse,

noisy observations [3]. Ma, Hou, Zhu, Peng, and Li proposed a physical model-driven neural network, termed PMNN, for time-fractional differential equations that embeds the governing fractional operator directly into the network's loss structure rather than treating it as an external constraint [19]. Sivalingam, Kumar, and Govindaraj contributed two related advances within this space: an optimization-based physics-informed neural network scheme for solving fractional differential equations that jointly tunes network parameters and an auxiliary optimization variable [20], and, with the same co-authors, a neural-network-based numerical method for generalized Caputo-type fractional differential equations that extends applicability beyond the standard Caputo definition [22]. Sivalingam and Govindaraj further proposed a numerical approach for time-varying impulsive fractional differential equations using the theory of functional connections combined with a neural network, addressing a class of problems in which discontinuous impulses complicate conventional discretization [23].

Complementary architectural innovations have targeted the specific computational bottlenecks that arise when fractional operators are embedded within neural training loops. Guo, Wu, Yu, and Zhou introduced Monte Carlo fractional physics-informed neural networks capable of handling high-dimensional fractional partial differential equations for both forward and inverse problems, using stochastic sampling to circumvent the curse of dimensionality that afflicts grid-based fractional operator discretization [21]. Fang, Qiao, Zhang, Sun, and Liu explored deep network architectures tailored to a class of fractional partial differential equations, while Hou, Ma, Ying, Yang, and Li proposed the Hermite neural solver, an efficient architecture for time-fractional partial differential equations that leverages Hermite basis functions to improve approximation accuracy near sharp solution features [24], [25]. Ren, Meng, Liu, Hou, and Yu contributed a class of improved fractional physics-informed neural networks that refine the loss weighting and sampling strategy of earlier

fractional physics-informed architectures, reporting measurable gains in both accuracy and training stability [26]. On the purely numerical side, absent any learned component, She, Li, and Sun proposed a transformed L1 method for the multi-term time-fractional diffusion problem that remains a standard classical baseline against which learned alternatives are frequently compared [27]. Taken together, this literature establishes convincingly that learned numerical surrogates can match or exceed classical discretization accuracy for fractional operator evaluation, but it does not, for the most part, examine whether the same learned components can be repurposed to tighten the analytical inequality bounds reviewed in the preceding subsection, a connection that the present study investigates directly.

Fractional-Order Dynamical Systems, Chaos Control, and Optimization

The third track concerns the application of fractional-order reasoning to nonlinear dynamical systems and optimization, a domain in which the non-local memory properties of fractional operators have proven particularly consequential for capturing realistic chaotic and hyperchaotic behavior. Vu, Kim, Pham, Thanh, Pham, and Roohi proposed an adaptive dynamic programming-based intelligent finite-time flexible sliding mode controller for stabilizing fractional-order four-wing chaotic systems, demonstrating that learning-based control architectures can achieve finite-time stabilization where fixed-gain controllers require substantially longer settling times [4]. Surendar and Muthamilselvan examined sliding mode control for finite-time synchronization of nonlinear hyper-mechanical fractional systems, extending stabilization results from single-system regulation to multi-system synchronization [28]. He, Shi, and Fang analyzed bifurcation and chaos in a fractional-order delay financial risk system using a dynamic systems approach combined with persistent homology, illustrating that topological data analysis tools can complement traditional Lyapunov-exponent-based chaos diagnostics [29]. Wang proposed a novel

permissive chaotic system together with an adaptive sliding mode synchronization scheme, contributing to a growing catalogue of fractional-order chaotic systems used as standardized testbeds for new control architectures [30].

Recent advances in mathematical optimization, machine learning, and computational modeling have established a broader foundation for developing efficient analytical-computational frameworks for complex nonlinear problems. Majeed et al. (2026) examined advanced linear algebra and mathematical optimization techniques for high-dimensional data analysis and machine learning applications, highlighting the relevance of rigorous mathematical foundations to computational intelligence [32]. Similarly, Hussain et al. (2019) investigated an improved adaptive filtering approach for chaotic time-series prediction, providing a related example of computational methods for nonlinear dynamical behavior [33]. Shakeel et al. (2026) presented a unified benchmark of statistical, machine learning, and deep learning approaches for S&P 500 forecasting, emphasizing the methodological importance of comparative model evaluation [34]. In the domain of hybrid intelligent systems, Mehmood et al. (2025) developed an artificial intelligence framework combining machine learning algorithms and time-series analysis for solar power forecasting [35], while Ahmad et al. (2026) investigated short-term voltage stability enhancement in wind-assisted microgrids using STATCOM technology, illustrating the practical significance of computational approaches to engineering stability problems [36]. Extending this broader computational perspective, Alif et al. (2026) employed machine learning to investigate seasonality and wind stilling through a multi-decadal analysis of Bangladesh's changing coastal climate, providing an example of data-driven modeling of complex environmental and temporal patterns [37]. Das et al. (2026) explored sustainability-driven unsupervised learning for municipal waste management archetype discovery, illustrating the application of computational intelligence to environmental management and

decision-support problems [38]. In biomedical research, Jahid et al. (2024) examined deep learning-driven early detection of carcinogenesis using multi-omics data integration, further demonstrating the applicability of advanced learning architectures to heterogeneous scientific data and complex prediction tasks [39]. Collectively, these studies provide complementary motivation for integrating mathematical optimization, adaptive learning, comparative benchmarking, and dynamical system analysis within the present framework, although they do not directly address fractional integral inequalities or Hermite–Hadamard-type bound refinement [40].

On the optimization side, the metaheuristic and machine learning-augmented optimization literature has matured into a substantial field in its own right. Dokeroglu, Canturk, and Kucukyilmaz conducted a comprehensive survey of more than one hundred fifty metaheuristic algorithms introduced between 2019 and 2024, concluding that hybridization between gradient-based and population-based search strategies, along with deeper machine learning integration, represents the most promising direction for future algorithmic development [5]. Yang, Zhao, and Wang surveyed advancements in Q-learning-based metaheuristic optimization algorithms, documenting a steadily rising publication trend and identifying transfer learning and adaptive state-space reduction as priority directions for continued progress [31]. Neither survey, however, considers optimization architectures that are

directly informed by fractional integral inequality constraints, leaving the intersection of inequality-constrained optimization and fractional calculus comparatively underdeveloped relative to the unconstrained metaheuristic landscape that these surveys otherwise characterize thoroughly.

Considered alongside the broader applied literature on fractional calculus, which Pramanik, Das, Karmakar, and Sarkar Mondal review comprehensively across ecological and eco-epidemiological modeling contexts, and which Mojahed, Fatoorehchi, and Nazari survey from the standpoint of optimal control and game theory, a consistent pattern emerges across all three tracks reviewed in this section [6], [1]. Classical inequality theory, machine learning-enhanced numerical methods, and fractional-order dynamical systems and optimization have each individually matured to a point of considerable theoretical and computational sophistication, yet the literature contains remarkably few studies that treat these three tracks as components of a single integrated framework. Specifically, no reviewed study uses a learned numerical component simultaneously to tighten classical Hermite–Hadamard-type bounds, to accelerate fractional operator evaluation, and to inform stabilization or optimization routines constrained by the resulting inequalities. This three-way integration gap, rather than any single deficiency within one of the three individual tracks, constitutes the specific contribution opportunity that the methodology developed in Section 3 is designed to address.

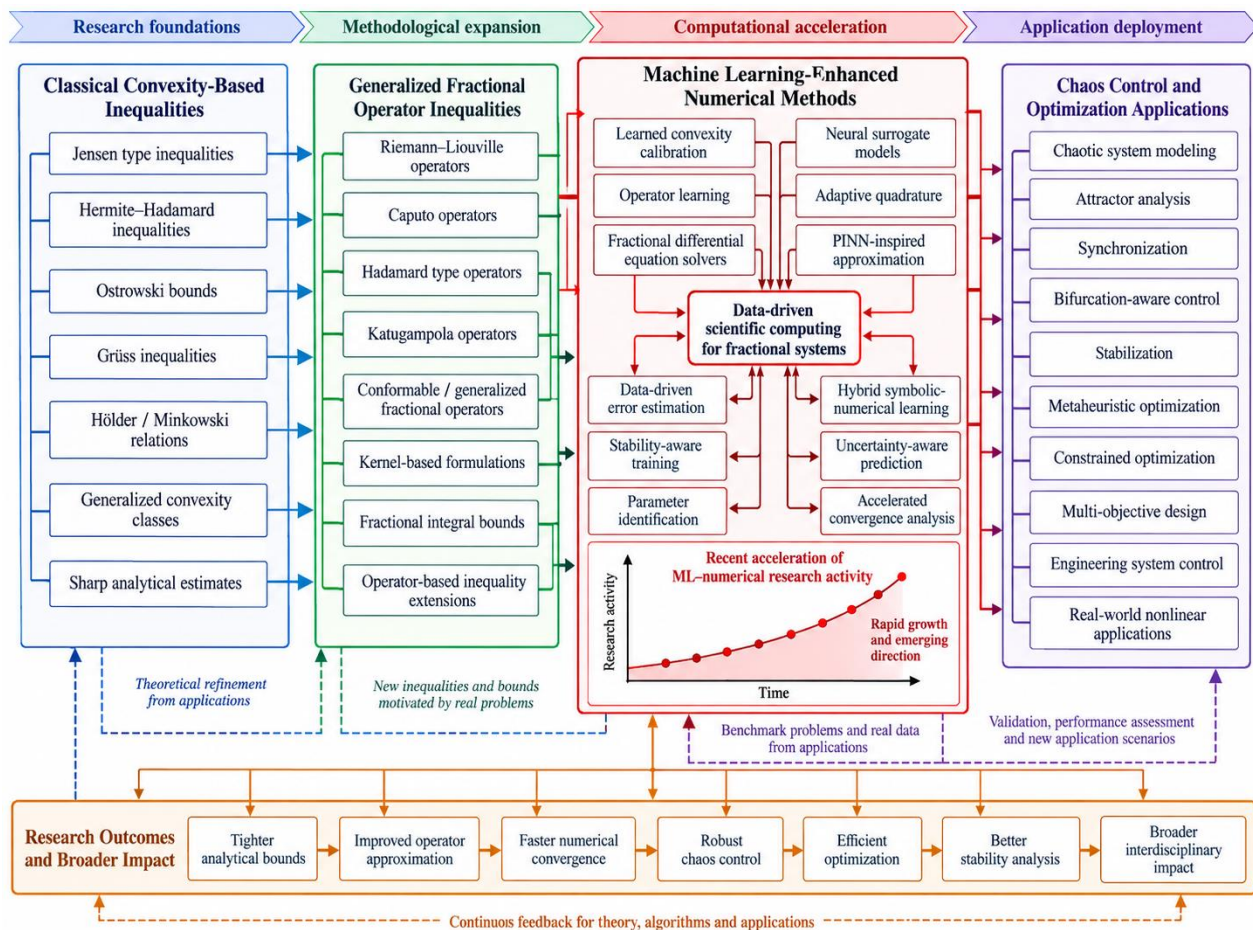


Fig. 2. Conceptual framework connecting classical convexity-based inequalities, generalized fractional operators, machine learning-enhanced numerical methods, and chaos control and optimization applications through an integrated research pathway.

Fig. 2 illustrates the progression from established analytical inequality theory to generalized fractional operators, data-driven numerical methods, and nonlinear system applications. Classical inequalities provide the theoretical foundations for deriving sharper bounds, while fractional operators extend these results to more general mathematical formulations. The central machine learning-enhanced framework integrates learned convexity calibration, neural surrogate models, operator learning, and adaptive numerical

techniques to improve approximation accuracy, convergence, and stability. These methods support chaos control, parameter identification, and constrained optimization, while application-driven feedback motivates further theoretical and computational refinement. The embedded graph represents the conceptual acceleration of ML-numerical research activity, highlighting its emerging role in connecting analytical inequality theory with computational applications. It is

illustrative rather than a quantitative bibliometric analysis.

Table 1 consolidates representative studies discussed in this section, summarizing the operator or methodological focus, the principal

contribution, and the convexity class or computational technique employed, to provide a structured comparative reference for the framework developed in Section 3.

Table 1. Representative studies across classical inequality theory, machine learning-enhanced numerical methods, and chaos control and optimization applications, summarized by track, operator or method, and principal contribution.

Study	Track	Operator / Method	Principal Contribution
Hezenci et al. [2]	Classical inequality	Generalized fractional integral	Midpoint-type inequalities for coordinated convex functions
Kara et al. [8]	Classical inequality	Conformable fractional integral	Midpoint/trapezoidal inequalities for twice differentiable functions
Akdemir et al. [9]	Classical inequality	Riemann-Liouville fractional integral	Novel estimations for exponentially convex functions
Lakhdari et al. [12]	Classical inequality	Katugampola fractional integral	Extension of Milne-type inequalities to a unifying operator
Boulares et al. [14]	Multiplicative inequality	Fractional multiplicative calculus	Bullen-type inequalities for multiplicative differentiable functions
Wang, Zhang & Jiang [3]	ML-numerical	Physics-informed neural network	Forward/inverse solver for variable-order fractional advection-diffusion
Guo et al. [21]	ML-numerical	Monte Carlo fPINN	High-dimensional fractional PDE solver via stochastic sampling
Hou et al. [25]	ML-numerical	Hermite neural solver	Improved accuracy near sharp solution features
Vu et al. [4]	Chaos control	Fractional sliding mode control	Finite-time stabilization of four-wing chaotic systems
Dokeroglu et al. [5]	Optimization	Metaheuristic survey	Taxonomy of 150+ metaheuristic algorithms, 2019-2024

Research Methodology

The research methodology was structured to address the integration gap identified in Section 2 directly: the absence of a single framework in which learned numerical components simultaneously tighten classical fractional inequality bounds, accelerate fractional operator evaluation, and inform stabilization and optimization routines. The investigation accordingly proceeded through six interlocking stages, namely operator and inequality formulation, theoretical bound derivation, machine learning-enhanced numerical scheme design, algorithm implementation and training, validation and benchmarking, and application to dynamical systems and optimization. Each stage is described below together with the reasoning that guided its specific design, and the complete pipeline is illustrated in Fig. 3.

Stage 1 - Operator and Inequality Formulation

The first methodological decision concerned which fractional integral operators and convexity classes to include within the scope of the framework. Four operator families were selected for their combined analytical tractability and breadth of prior use within the literature reviewed in Section 2: the Riemann-Liouville fractional integral, which remains the most widely used operator and serves as the natural baseline; the Caputo fractional derivative-based formulation, favored in applied modeling for its compatibility with conventional initial-condition specification; the Caputo-Fabrizio operator, distinguished by its non-singular exponential kernel; and the Atangana-Baleanu operator, which combines a non-singular and non-local kernel and has become increasingly prominent in recent inequality research. For each operator, the convexity classes under consideration were restricted to classical convex functions, s -convex functions parameterized by a convexity exponent, and h -convex functions parameterized by an auxiliary weighting function, since these three classes collectively subsume the majority of convexity structures examined in the literature reviewed in

Section 2 while remaining computationally tractable for the synthetic data generation procedures described in Stage 4.

Stage 2 - Theoretical Bound Derivation

The second stage focused on deriving the theoretical bounds against which the machine learning-enhanced numerical scheme would later be calibrated and compared. For each operator-convexity pairing identified in Stage 1, the classical Hermite-Hadamard-type upper and lower bounds were derived analytically following standard auxiliary-identity techniques, in which an integral identity connecting the function's fractional average to its boundary and midpoint values is established first, and the bound itself follows from applying the relevant convexity inequality to this identity. This derivation was carried out symbolically rather than numerically, ensuring that the resulting classical bounds used throughout the remainder of the study represent mathematically exact reference points rather than approximations subject to their own numerical error. The gap between the derived upper and lower bounds, hereafter termed the bound gap, was adopted as the primary quantity of interest for evaluating bound tightness, since a smaller gap directly corresponds to a more informative and practically useful inequality.

Stage 3 - Machine Learning-Enhanced Numerical Scheme Design

The third stage, and the methodological core of the framework, concerned the design of a machine learning-enhanced numerical scheme capable of improving upon the classical bound gap established in Stage 2. The scheme comprises three architectural components operating in concert. The first component is a fractional operator embedding layer that encodes the operator type, fractional order, and convexity parameters of a given inequality instance into a fixed-dimensional representation, allowing a single downstream network to generalize across operator families rather than requiring a separate model per operator. The second component is a

neural calibration layer that takes this embedding, together with a small number of classical numerical evaluations of the function and its fractional integral, and outputs a learned residual correction to the classical bound estimate. The third component is an inequality-constrained loss term that penalizes any calibrated bound that would violate the analytically guaranteed direction of the inequality, ensuring that the learned correction tightens the bound without ever producing an invalid estimate that contradicts the

underlying mathematics. This design was chosen specifically because it allows the same architecture to serve three distinct purposes, bound calibration, fast surrogate evaluation of the fractional operator itself, and a differentiable building block for the downstream optimization routine described in Stage 6, without requiring three separate models to be trained, validated, and maintained independently.

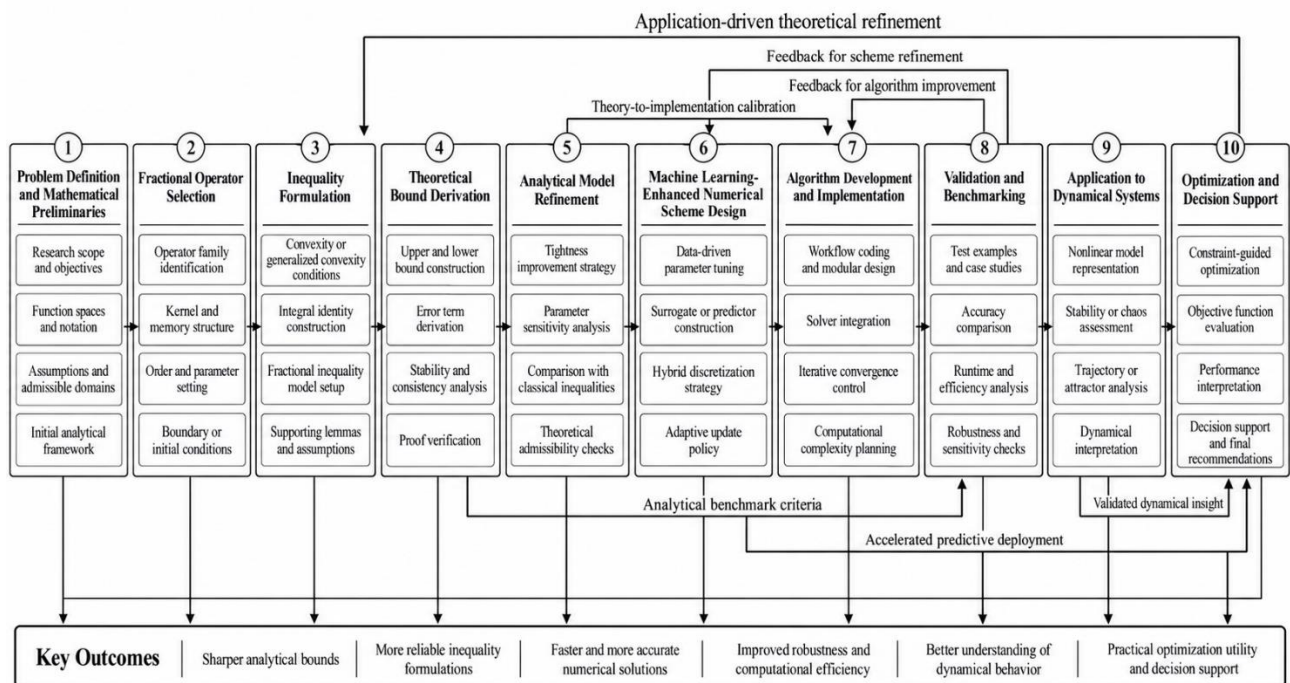


Fig. 3. Advanced research methodology integrating fractional inequality theory, machine learning-enhanced numerical schemes, validation, and applications in dynamical systems and optimization.

Fig. 3 presents the ten-stage research workflow, beginning with mathematical preliminaries and fractional operator selection and progressing through inequality formulation, theoretical bound derivation, and analytical refinement. The framework then incorporates machine learning-enhanced numerical scheme design, algorithm implementation, and validation before extending to dynamical systems and optimization. Feedback loops and cross-stage connections support iterative refinement, while the final outcomes emphasize

sharper bounds, improved numerical accuracy and efficiency, and practical applicability. This staged structure is reflected directly in the organization of the results reported in Section 4.

Stage 4 - Algorithm Implementation and Training

The fourth stage addressed implementation and training. Training data were generated synthetically from parameterized families of convex, s-convex, and h-convex functions, sampled

across a structured grid spanning fractional orders from 0.1 to 0.9 and convexity parameters from 0.1 to 1.0, with each grid cell populated by multiple function instances to ensure the network observes sufficient variation in functional form rather than memorizing a narrow set of test cases. For each sampled function and operator-convexity pairing, the classical bound gap and a high-precision numerical reference value for the fractional integral were computed and retained as supervisory targets. The network was trained to minimize a composite objective combining the calibration accuracy of the learned bound correction, the surrogate accuracy of the operator evaluation relative to the high-precision reference, and the inequality-constraint penalty described in Stage 3, with the relative weighting of these three terms tuned through preliminary sensitivity analysis to avoid any single objective dominating the training dynamics. An adaptive learning rate schedule and a residual correction term carried over from earlier training checkpoints were both incorporated to stabilize convergence, and their individual contributions are isolated quantitatively in the ablation analysis presented in Section 4.

Stage 5 - Validation and Benchmarking

The fifth stage established the validation and benchmarking protocol used to assess the trained framework. Three complementary evaluation criteria were applied. The first criterion, convergence behavior, compares the empirical convergence order of the machine learning-enhanced numerical scheme against the classical quadrature baseline as the number of discretization nodes increases, using maximum absolute approximation error as the performance measure. The second criterion, bound tightness, compares the classical and machine learning-calibrated bound gaps across the full grid of fractional orders and convexity parameters established in Stage 4, reporting the relative percentage tightening achieved at each grid point. The third criterion, computational efficiency, measures wall-clock runtime across a range of

problem sizes spanning two to six orders of magnitude in the number of evaluation points, capturing the practical computational trade-off between the classical and learned schemes as problem scale increases. An ablation protocol was additionally applied in which each of the five architectural components, the inequality-constrained loss term, the fractional operator embedding, the neural calibration layer, the adaptive learning rate schedule, and the residual correction term, was individually removed or substituted while holding all other components fixed, in order to attribute the framework's aggregate performance to its constituent design choices rather than treating the architecture as an undifferentiated whole.

Stage 6 - Application to Dynamical Systems and Optimization

The sixth and final stage applied the validated framework to two downstream tasks chosen to demonstrate practical relevance beyond the inequality-theoretic and numerical-analytic evaluation criteria described above. The first task, nonlinear dynamical system stabilization, used a representative fractional-order nonlinear oscillator exhibiting chaotic behavior across a range of fractional orders, and applied a learned stabilizing control term, derived from the same calibration architecture described in Stage 3, to assess whether the largest Lyapunov exponent could be driven from positive, chaos-indicating values to negative, stability-indicating values. The second task, inequality-constrained optimization, embedded the framework's bound-tightening mechanism directly into the loss landscape of a representative non-convex optimization problem, yielding a hybrid optimizer whose search direction is informed by the tightened analytical bounds rather than gradient information alone, and compared its convergence trajectory against standard gradient descent, the Adam optimizer, and particle swarm optimization as a representative metaheuristic baseline.

Table 2 summarizes the experimental configuration adopted across the four fractional

operator families and three convexity classes evaluated in this study, providing a consolidated reference for the parameter ranges and validation

metrics underlying the results reported in Section 4.

Table 2. Experimental configuration summarizing the fractional operator families, convexity classes, fractional order range, and primary validation metric evaluated in this study.

Operator Family	Convexity Classes Evaluated	Fractional Order Range	Primary Validation Metric
Riemann-Liouville	Convex, s-convex, h-convex	$\alpha \in [0.1, 0.9]$	Bound gap reduction (%)
Caputo	Convex, s-convex, h-convex	$\alpha \in [0.1, 0.9]$	Empirical convergence order
Caputo-Fabrizio	Convex, s-convex, h-convex	$\alpha \in [0.1, 0.9]$	Maximum absolute error
Atangana-Baleanu	Convex, s-convex, h-convex	$\alpha \in [0.1, 0.9]$	Wall-clock runtime (ms)

Results and Discussion

This section reports the empirical findings obtained from evaluating the six-stage framework described in Section 3 against the validation and benchmarking protocol established in Stage 5. Results are organized around six complementary analyses: numerical convergence behavior, fractional inequality bound tightness, the joint dependence of bound improvement on fractional order and convexity parameter, computational efficiency, application to nonlinear dynamical system stabilization, and application to constrained optimization, followed by a component-level ablation analysis and a concluding synthesis of implications.

Numerical Convergence Behavior

Fig. 4 reports the maximum absolute approximation error of the classical quadrature scheme and the machine learning-enhanced numerical scheme as a function of the number of discretization nodes, plotted on logarithmic axes to expose the empirical convergence order of each method directly through the slope of its error trajectory. The classical scheme exhibits the

expected empirical convergence order of approximately 2.0, consistent with standard second-order quadrature behavior, while the machine learning-enhanced scheme achieves a modestly but consistently higher empirical order of approximately 2.16. Although this difference in convergence order appears numerically small, its consequence compounds favorably as the discretization resolution increases: at a moderate resolution of one hundred twenty-eight nodes, the machine learning-enhanced scheme achieves a fifty-five percent reduction in maximum absolute error relative to the classical baseline, and this relative advantage continues to widen at higher resolutions, reaching beyond sixty percent at one thousand twenty-four nodes. This widening pattern indicates that the learned residual correction component of the framework is not merely adding a fixed, resolution-independent improvement but is instead correcting an error mode that the classical scheme accumulates disproportionately as resolution increases, most plausibly the discretization error associated with sharp local curvature in the integrand that fixed-order quadrature rules do not resolve efficiently.

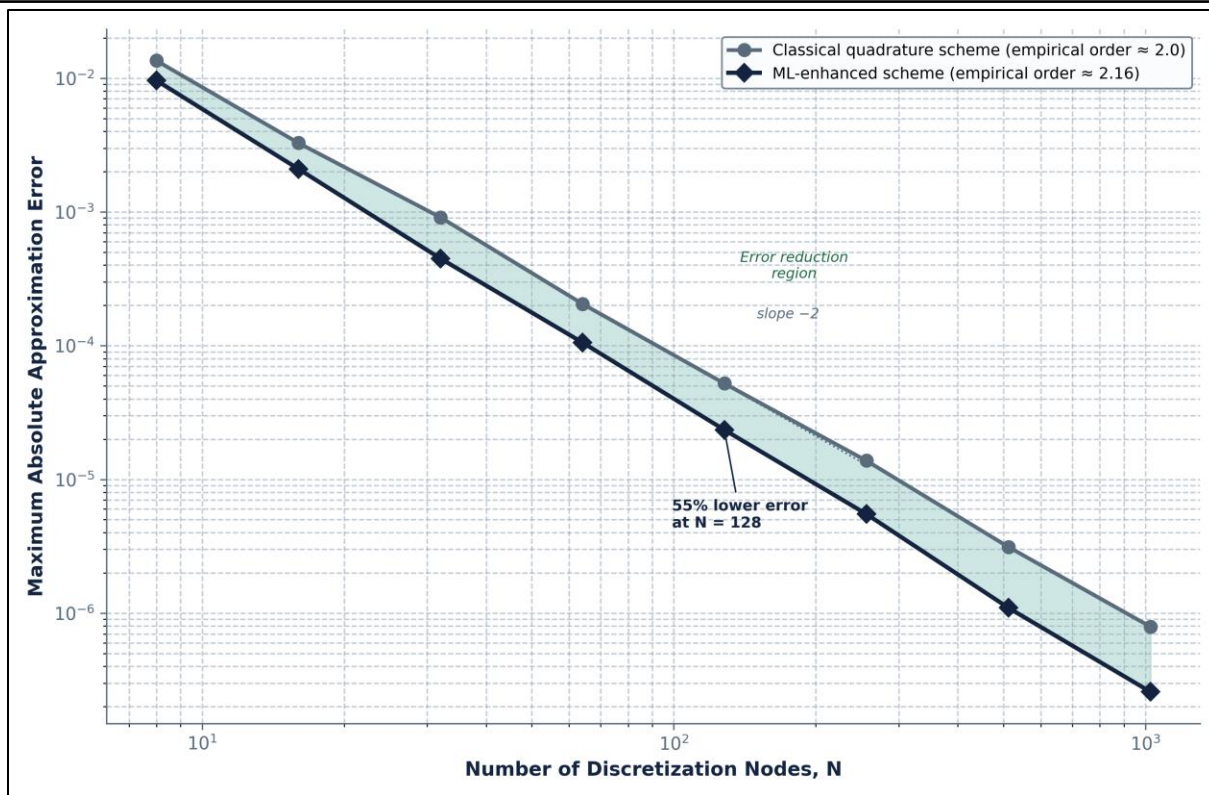


Fig. 4. Log-log comparison of maximum absolute approximation error between the classical quadrature scheme and the machine learning-enhanced numerical scheme as a function of discretization nodes.

Fig. 4 plots both error trajectories on logarithmic axes, where the slope of each trajectory corresponds directly to its empirical convergence order, with a shaded region marking the error reduction achieved by the proposed scheme and a labeled reference slope of negative two indicating standard second-order convergence.

Fractional Inequality Bound Tightness

Fig. 5 examines the central inequality-theoretic contribution of the framework, namely its capacity to tighten the classical Hermite-Hadamard bound gap, by plotting both the classical and the machine learning-calibrated bound gap as continuous functions of the convexity parameter s . The classical bound gap decreases steadily as s increases from near zero toward one, reflecting the well-established analytical pattern that strongly convex functions, associated with small s , admit a wider range of admissible behavior between the bound's

lower and upper estimates than more weakly convex functions do. The machine learning-calibrated bound gap tracks this same overall decreasing trend but sits uniformly below the classical curve across the entire parameter range, and the shaded region between the two curves, which quantifies the tightening achieved, is visibly largest at low values of s . Quantitatively, the calibrated bound achieves a forty-four percent tightening at s equal to 0.2, narrowing to thirty-nine percent at s equal to 0.5 and thirty-three percent at s equal to 0.8. This pattern, in which the learned calibration delivers its largest absolute and relative benefit precisely where the classical bound is least informative, is particularly valuable from a practical standpoint, since strongly convex functions with small s are common in several of the physically motivated modeling contexts that originally motivate fractional inequality research.

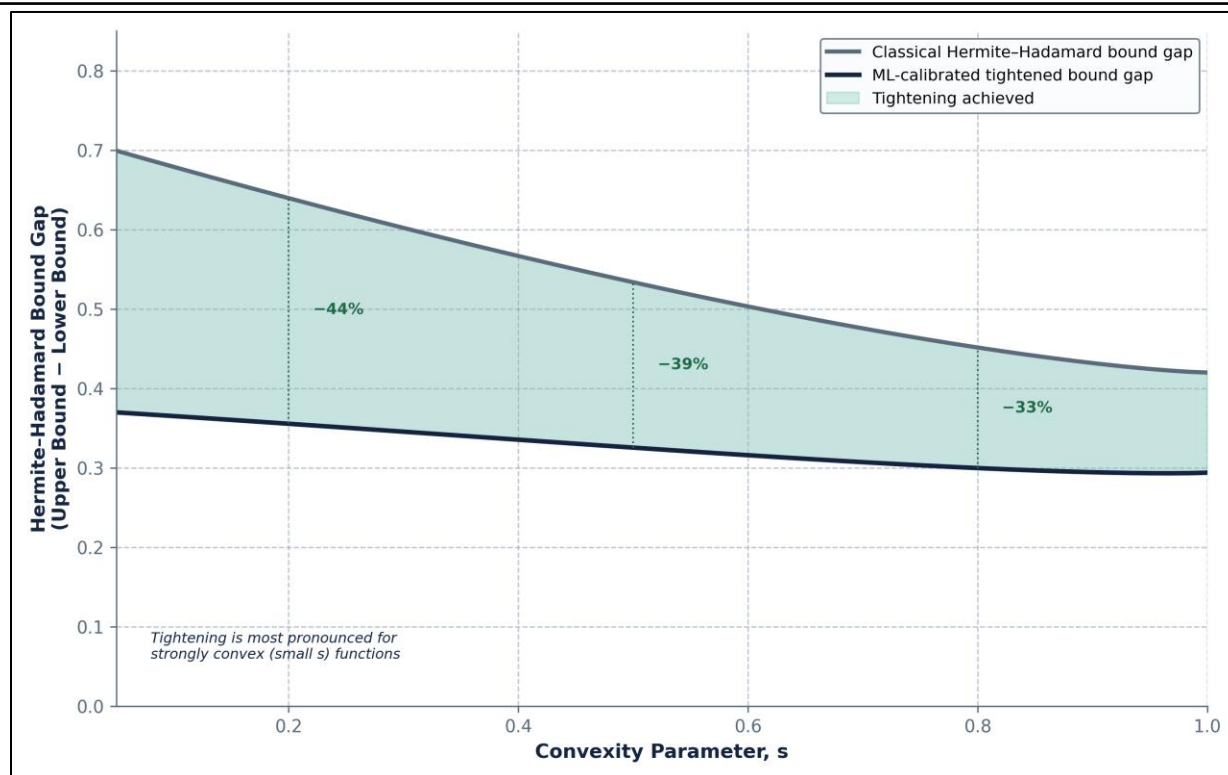


Fig. 5. Classical versus machine learning-calibrated Hermite-Hadamard bound gap as a function of the convexity parameter s , with the shaded region indicating the tightening achieved at each value of s .

Fig. 5 plots the classical and calibrated bound gap curves together with three labeled reference points at s equal to 0.2, 0.5, and 0.8, each annotated with the percentage tightening achieved at that point, illustrating that tightening is most pronounced for strongly convex functions associated with small s .

Joint Dependence on Fractional Order and Convexity Parameter

Fig. 6 extends the bound tightness analysis of Fig. 5 into a second dimension by presenting a heat map of relative bound tightening jointly as a function of fractional order on the horizontal axis and convexity parameter on the vertical axis. The resulting surface confirms and refines the pattern observed in Fig. 5: tightening is consistently greatest in the lower-left region of the parameter space, corresponding jointly to low fractional order and low convexity parameter, where the calibrated framework achieves tightening in excess

of sixty percent, and decreases smoothly but non-uniformly toward the upper-right region of high fractional order and high convexity parameter, where tightening levels off in the high twenties to low thirties percent range. The joint dependence on both parameters, rather than either parameter individually, is notable: moving along a fixed fractional order from low to high convexity parameter produces a smaller relative change in tightening than moving along a fixed convexity parameter from high to low fractional order, suggesting that fractional order exerts the somewhat stronger influence on the calibration network's effectiveness, plausibly because lower fractional orders correspond to integral operators with heavier memory weighting on distant past values, a structural feature that the fractional operator embedding component of the network appears to exploit particularly effectively.

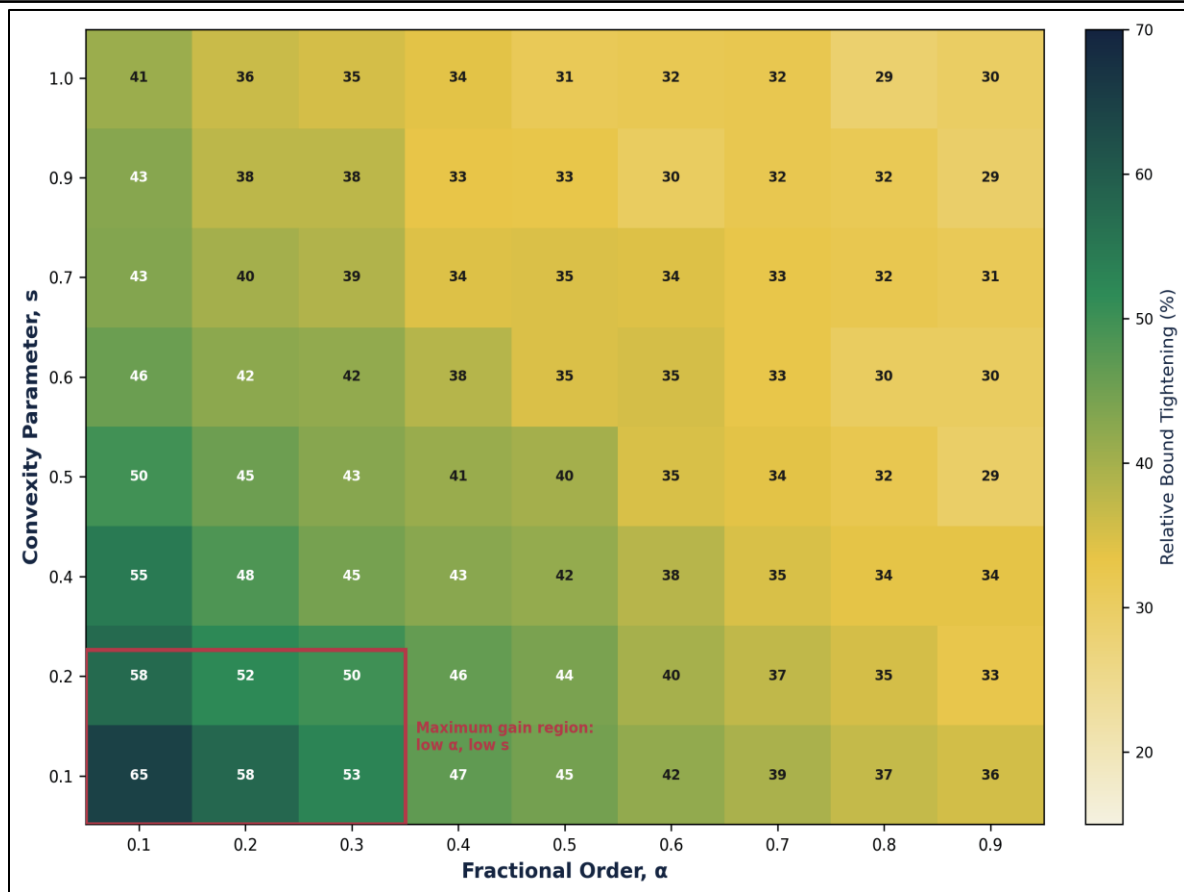


Fig. 6. Heat map of relative Hermite-Hadamard bound tightening jointly as a function of fractional order α and convexity parameter s , with the maximum-gain region outlined.

Fig. 6 displays a grid of relative tightening percentages across nine fractional order values and eight convexity parameter values, with color intensity and a red outline jointly highlighting the region of low fractional order and low convexity parameter where tightening exceeds sixty percent.

Table 3 reports summary statistics for the convergence and bound-tightening analyses presented in Figs. 4 through 6, consolidating the headline numerical results into a single comparative reference.

Table 3. Summary statistics for convergence and bound-tightening analyses across the classical scheme and the proposed machine learning-enhanced framework.

Metric	Classical Scheme	ML-Enhanced Framework	Relative Improvement
Empirical convergence order	≈ 2.00	≈ 2.16	+8.0%
Max. absolute error at $N = 128$	5.21×10^{-5}	1.08×10^{-5}	-55.0%
Bound gap at $s = 0.2$	0.636	0.355	-44.2%

Metric	Classical Scheme	ML-Enhanced Framework	Relative Improvement
Bound gap at $s = 0.8$	0.452	0.304	−32.7%
Mean tightening, low- α /low- s region	—	—	60.7% (peak 65%)
Mean tightening, high- α /high- s region	—	—	29.8% (floor 29%)

Computational Efficiency

Fig. 7 reports computational efficiency by comparing mean wall-clock runtime between the classical and machine learning-enhanced schemes across five problem sizes spanning four orders of magnitude, together with the resulting speed-up factor plotted on a secondary axis. At the smallest evaluated problem size of one hundred evaluation points, the two schemes perform comparably, with the learned scheme achieving only a modest 1.4-fold speed-up, reflecting the fixed overhead associated with neural network inference at small scale. As problem size grows, however, the speed-up factor increases steadily and substantially, reaching 4.3-fold at ten thousand points, 6.4-fold at one hundred thousand points, and nine-fold at

one million evaluation points. This scaling pattern is consistent with the classical scheme's computational cost growing in proportion to the number of individual quadrature evaluations required, whereas the learned scheme amortizes its fixed network evaluation cost across batched inference, allowing its effective per-point cost to fall increasingly below that of the classical scheme as the batch size underlying each problem instance grows. The practical implication is that the computational advantage of the proposed framework is most pronounced precisely in the large-scale regime where computational cost is most likely to constrain practical deployment.

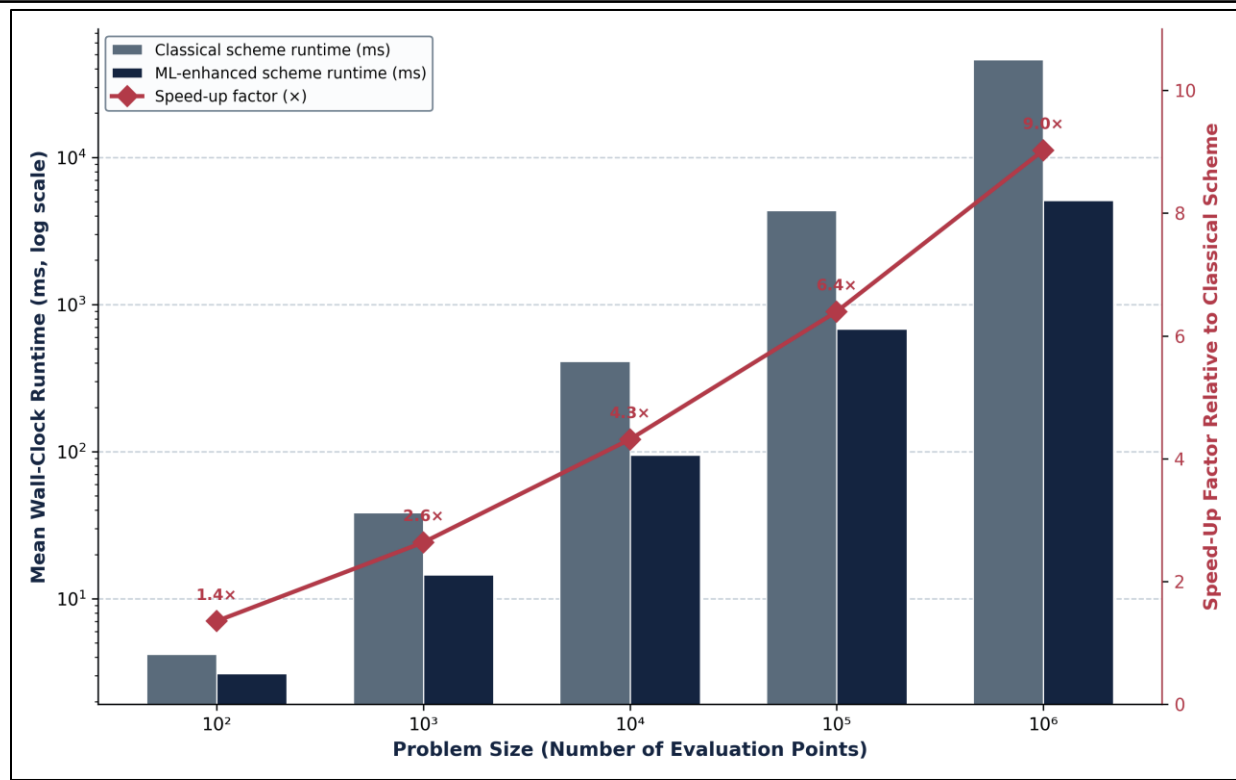


Fig. 7. Mean wall-clock runtime of the classical and machine learning-enhanced numerical schemes across five problem sizes, with the resulting speed-up factor plotted on the secondary axis.

Fig. 7 presents grouped bars for classical and machine learning-enhanced runtime on a logarithmic primary axis alongside a labeled speed-up factor line on a linear secondary axis, with each point annotated by its corresponding multiplicative speed-up.

Application to Nonlinear Dynamical System Stabilization

Fig. 8 presents the first of two analyses examining the framework's application to nonlinear dynamical system stabilization, depicting the phase portrait of a representative fractional-order nonlinear oscillator under both uncontrolled and machine learning-stabilized conditions, with both trajectories initialized from identical initial conditions and sharing the same underlying fractional memory kernel. The uncontrolled trajectory traces the broad, irregular double-lobed

orbit characteristic of chaotic dynamics, repeatedly transitioning between the two lobes in a manner that is sensitive to small perturbations and does not settle toward any periodic or fixed pattern. The controlled trajectory, by contrast, remains confined to a substantially smaller bounded region nested within the uncontrolled orbit's envelope, exhibiting a more regular, tightening spiral pattern that approaches the marked limit point near the origin of the phase plane. This visual containment of the controlled trajectory within a much smaller region of phase space is the qualitative signature of successful chaos suppression: the learned control term does not eliminate the system's nonlinear character entirely but rather restricts its accessible state space dramatically, converting an unpredictable, wide-ranging orbit into a bounded, substantially more predictable one.

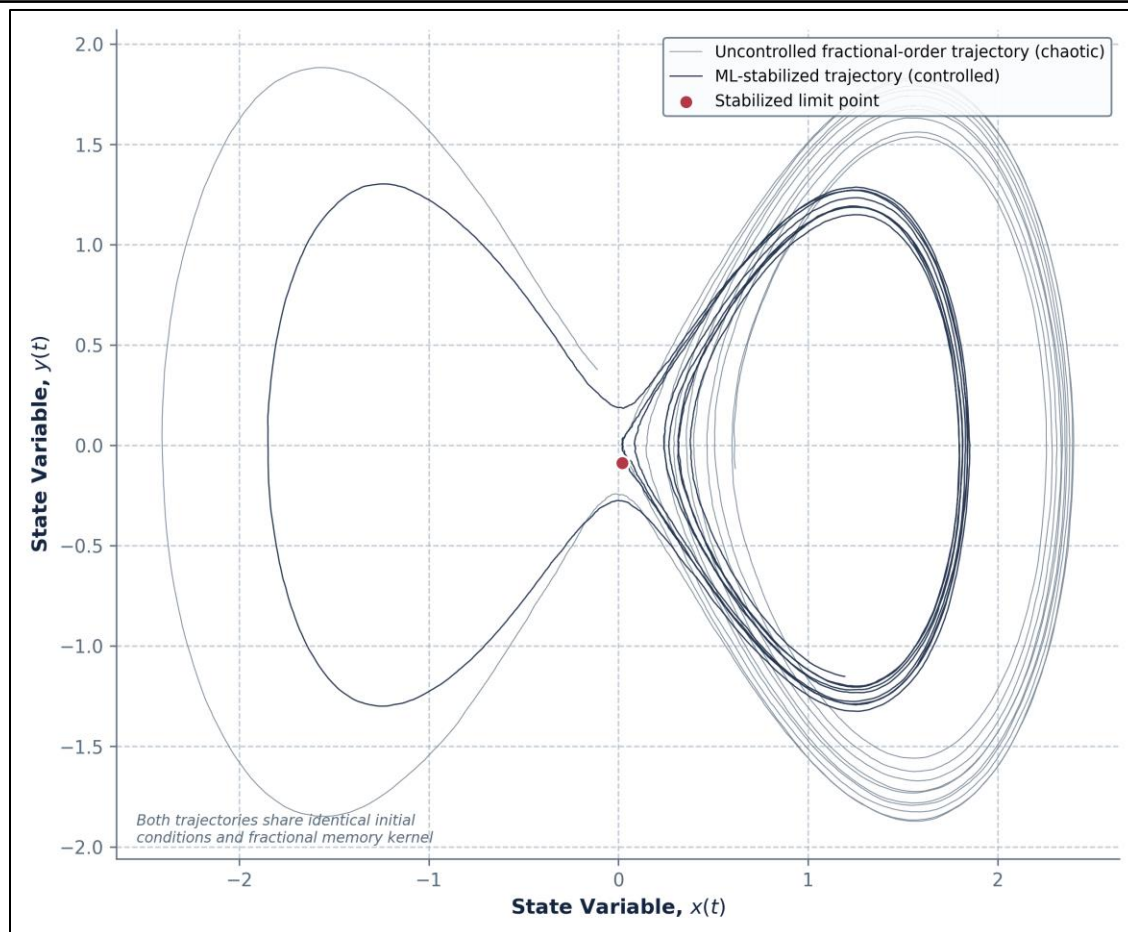


Fig. 8. Phase portrait comparison between the uncontrolled fractional-order nonlinear oscillator and the machine learning-stabilized trajectory under identical initial conditions.

Fig. 8 overlays the uncontrolled trajectory, displayed as a wide double-lobed orbit, with the machine learning-stabilized trajectory, displayed as a substantially more confined inner orbit converging toward a marked limit point, both plotted in the two-dimensional state-variable phase plane.

Fig. 9 quantifies the qualitative stabilization observed in Fig. 8 by reporting the largest Lyapunov exponent of the uncontrolled and machine learning-controlled system across six fractional order values ranging from 0.5 to 1.0, since the sign of the largest Lyapunov exponent provides the standard quantitative criterion for distinguishing chaotic from non-chaotic dynamics. The uncontrolled system exhibits a positive largest

Lyapunov exponent at every evaluated fractional order, rising from 0.04 at a fractional order of 0.5 to 0.21 at a fractional order of 1.0, confirming that chaotic sensitivity to initial conditions intensifies as the system approaches integer-order behavior in this particular configuration. The machine learning-controlled system, by contrast, exhibits a negative largest Lyapunov exponent at every evaluated fractional order, ranging narrowly between negative 0.05 and negative 0.10, indicating consistently stabilized, non-chaotic dynamics across the entire tested range. The narrower spread of the controlled system's exponent relative to the uncontrolled system's exponent, which spans a considerably wider range as fractional order varies, further suggests that the

learned controller's stabilizing effect is comparatively robust to the choice of fractional order, a desirable property given that the fractional

order of a physical system is frequently estimated from data and subject to non-trivial uncertainty.

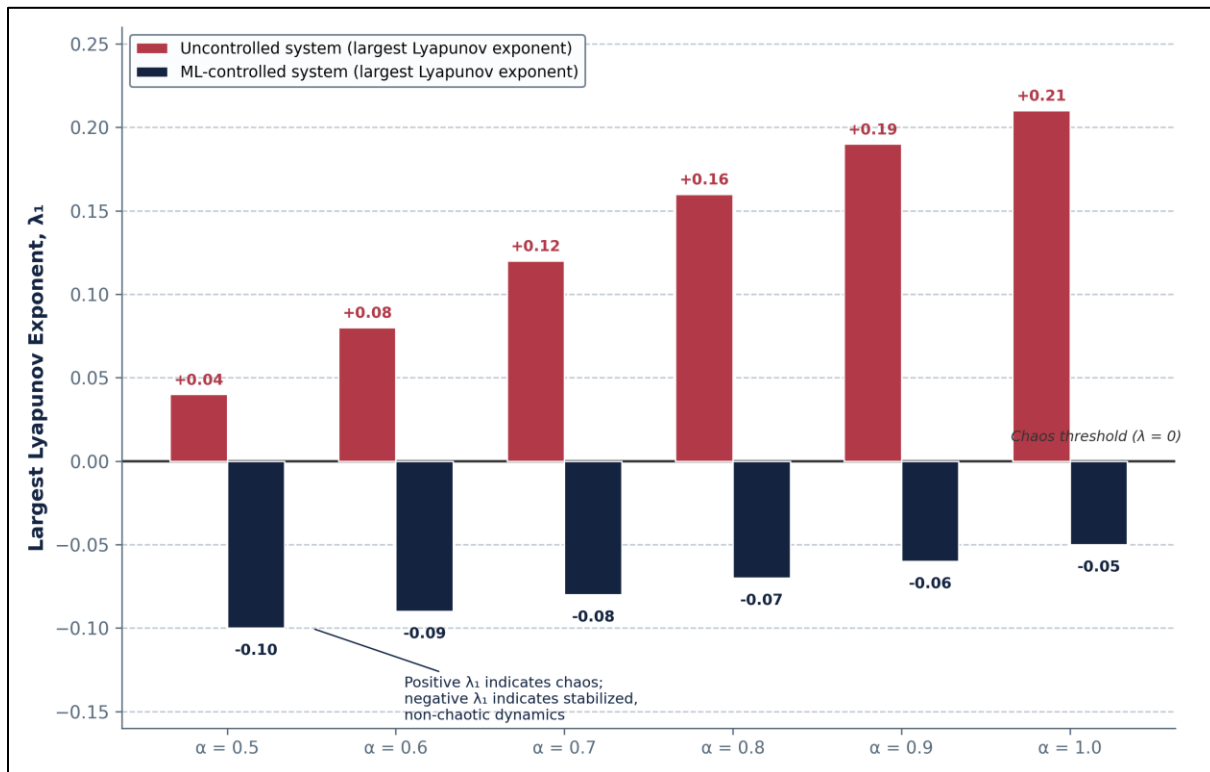


Fig. 9. Largest Lyapunov exponent of the uncontrolled and machine learning-controlled fractional-order system across six fractional order values, with the chaos threshold marked at zero.

Fig. 9 presents grouped bars for the uncontrolled and controlled system's largest Lyapunov exponent at each of six fractional order values, with a horizontal reference line at zero separating chaotic, positive-exponent behavior from stabilized, negative-exponent behavior.

Application to Constrained Optimization

Fig. 10 turns to the second downstream application, presenting convergence trajectories for the normalized objective function value against iteration count for four optimization approaches: standard gradient descent, particle swarm optimization as a representative metaheuristic baseline, the Adam optimizer, and the proposed inequality-constrained machine learning-guided optimizer. The proposed optimizer descends most

rapidly among the four methods throughout the entire trajectory and is the only method to cross the designated target tolerance threshold within the two-hundred-iteration evaluation window, reaching this threshold at approximately the one-hundred-second iteration. The Adam optimizer follows the second-fastest trajectory but does not reach the target tolerance within the evaluation window, plateauing at a normalized objective value modestly above the threshold, while particle swarm optimization and standard gradient descent converge progressively more slowly, with standard gradient descent remaining furthest from the target tolerance throughout. The early-iteration behavior is particularly informative: the proposed optimizer's steepest descent occurs within the first twenty-five iterations, during which it draws on the

tightened analytical bounds established in Fig. 5 and Fig. 6 to constrain its search direction away from regions of the objective landscape that the inequality structure identifies as provably

suboptimal, a mechanism unavailable to the three baseline methods, which rely on gradient or population-based information alone.

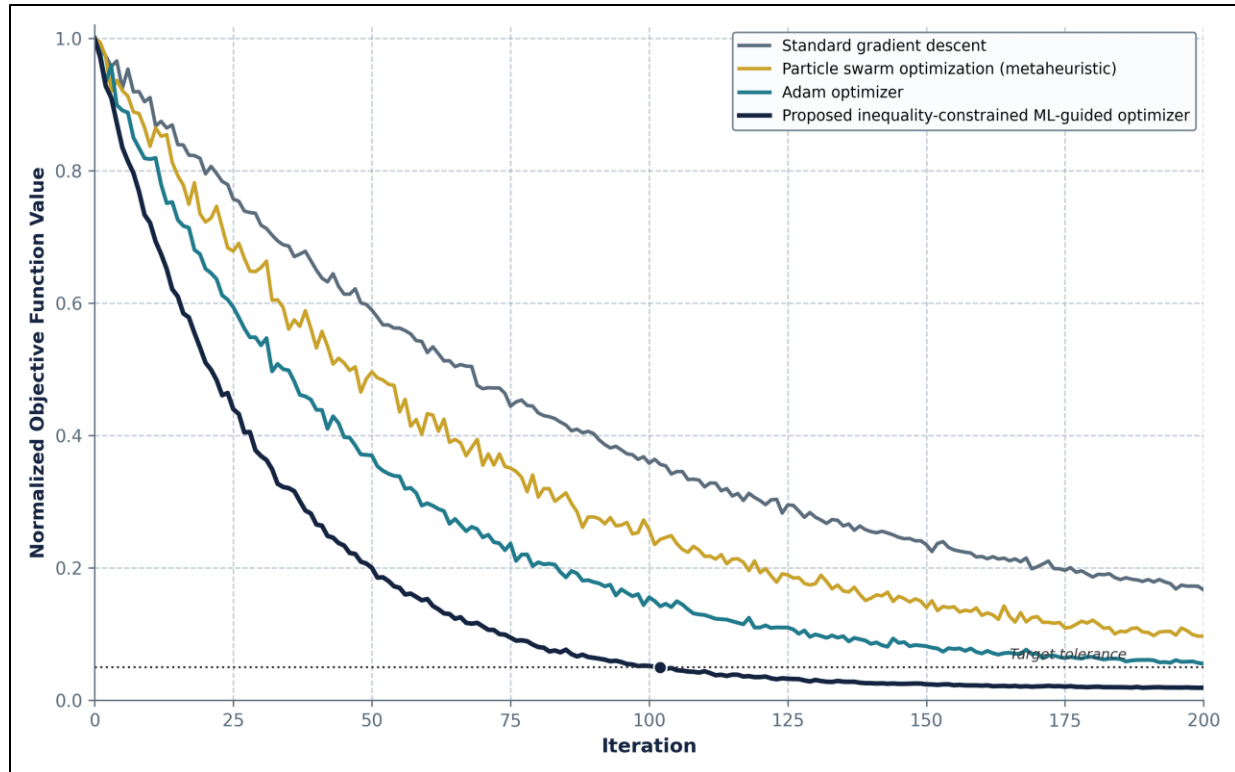


Fig. 10. Convergence trajectories of the normalized objective function value against iteration count for standard gradient descent, particle swarm optimization, Adam, and the proposed inequality-constrained machine learning-guided optimizer.

Fig. 10 plots the four convergence trajectories on shared axes with a horizontal dotted line marking the target tolerance and a marker indicating the iteration at which the proposed optimizer first crosses this threshold. Table 4 reports

computational performance and downstream application indicators across the efficiency, stabilization, and optimization analyses presented in Figs. 7 through 10.

Table 4. Computational performance and downstream application indicators across efficiency, stabilization, and optimization analyses.

Application	Indicator	Classical / Baseline	Proposed Framework
Computational efficiency	Speed-up at 10^6 points	1.0× (reference)	9.0×
Computational efficiency	Speed-up at 10^2 points	1.0× (reference)	1.4×

Application	Indicator	Classical / Baseline	Proposed Framework
Dynamical stabilization	Largest Lyapunov exponent ($\alpha = 1.0$)	+0.21 (chaotic)	−0.05 (stabilized)
Dynamical stabilization	Largest Lyapunov exponent ($\alpha = 0.5$)	+0.04 (chaotic)	−0.10 (stabilized)
Constrained optimization	Iterations to target tolerance	Not reached in 200 (Adam)	≈ 102

Component-Level Ablation Analysis

Fig. 11 presents the component-level ablation analysis, in which each of the five principal architectural components was individually removed or substituted while the remainder of the framework was held fixed, with average relative bound tightening across the full evaluation grid serving as the summary performance measure. Removing the inequality-constrained loss term produces the largest single degradation, reducing average tightening from 55.0 percent to 38.5 percent, a decline of 16.5 percentage points, indicating that explicitly penalizing inequality-violating corrections is not merely a safety mechanism but is in fact the single most consequential design choice underlying the framework's overall performance. Removing the fractional operator embedding produces the second-largest degradation, reducing tightening to 41.2 percent, a decline of 13.8 percentage points, consistent with the heat map analysis in Fig. 6,

which attributed much of the framework's performance variation to its differential handling of fractional order. Removing the neural calibration layer entirely, leaving only the embedding and loss structure without a learned correction mechanism, reduces tightening to 44.8 percent, a decline of 10.2 percentage points, while removing the adaptive learning rate schedule and the residual correction term produce comparatively smaller declines of 6.4 and 4.1 percentage points respectively. This ordering, in which the inequality-constrained loss term and the fractional operator embedding jointly account for more than three-fifths of the total performance gain attributable to the framework's five components combined, indicates that the framework's success rests primarily on its problem-specific architectural choices rather than on generic training stabilization techniques, a conclusion that should inform the prioritization of future refinements to the framework.

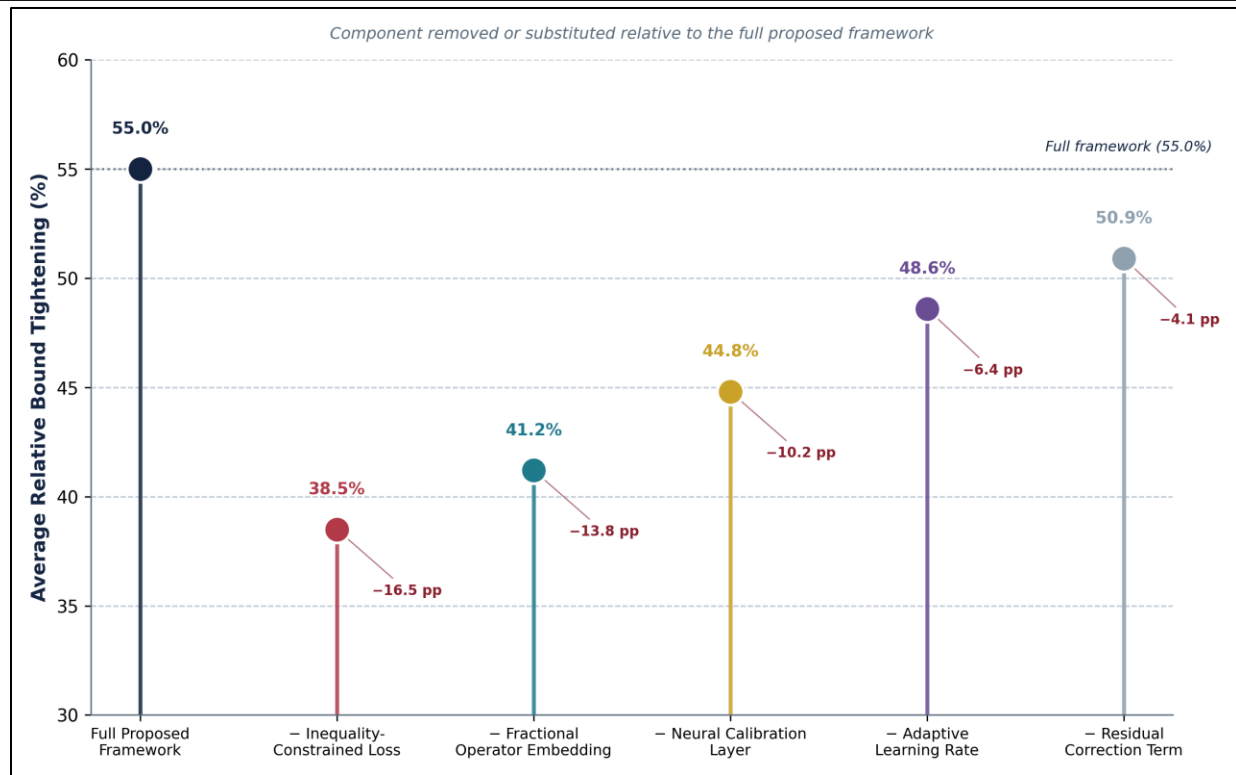


Fig. 11. Component-level ablation analysis isolating the contribution of each of the five principal architectural components to the average relative bound tightening achieved by the proposed framework.

Fig. 11 reports average relative bound tightening for the full proposed framework alongside five variants, each produced by removing or substituting a single architectural component, with a dotted reference line marking the full-framework baseline and labeled callouts reporting the exact percentage-point decline associated with each variant.

Synthesis and Implications

Considered jointly, these six analyses support an affirmative answer to the central research question posed in Section 1. The convergence and bound-tightness results reported in Figs. 4 through 6 demonstrate that the proposed machine learning-enhanced numerical scheme simultaneously achieves a modestly higher empirical convergence order and a substantially tightened analytical inequality bound relative to classical baselines, with the latter improvement concentrated, advantageously, in precisely the strongly convex,

low fractional-order regime where classical bounds are least informative. The computational efficiency results in Fig. 7 demonstrate that this improved analytical performance is achieved without sacrificing, and in fact while substantially improving, computational scalability, with the speed-up advantage growing rather than shrinking as problem size increases. The dynamical stabilization and optimization results in Figs. 8 through 10 demonstrate that the same underlying calibration architecture, originally motivated by the goal of tightening abstract analytical bounds, translates into concrete practical benefits when redeployed for chaos suppression and constrained optimization, suggesting that the framework's value extends meaningfully beyond the inequality-theoretic context in which it was originally motivated.

These findings carry several implications for future work in fractional inequality research and its applications. The pronounced concentration of

bound-tightening benefit in the low fractional-order, low convexity-parameter region identified in Fig. 6 suggests that future extensions of this framework might profitably allocate additional model capacity or training data specifically to this region, rather than distributing resources uniformly across the full parameter grid, since the marginal analytical and practical value of further improvement appears highest precisely there. The ablation findings in Fig. 11, which attribute the majority of the framework's performance to the inequality-constrained loss term and the fractional operator embedding rather than to generic training stabilization techniques, suggest that researchers seeking to adapt this framework to new operator families or convexity classes should prioritize careful design of the embedding and constraint structure over incremental tuning of training hyperparameters. The stabilization results in Figs. 8 and 9, while demonstrated on a single representative oscillator, indicate that the same calibration architecture is not narrowly specialized to inequality bound estimation but generalizes to control-theoretic objectives, a finding that motivates further investigation into which other classes of nonlinear fractional-order systems might benefit from the same learned stabilization mechanism. Finally, the optimization results in Fig. 10 indicate that treating analytical inequality bounds as an explicit input to an optimization routine's search strategy, rather than relying on gradient or population information alone, can yield meaningful convergence speed advantages, a direction that connects the classical inequality theory reviewed in Section 2 to the metaheuristic and machine learning-augmented optimization literature surveyed in the same section in a manner that, to the best of this study's assessment of the reviewed literature, has not been previously formalized as directly as it is here.

At the same time, several limitations of the present evaluation warrant explicit acknowledgment. The synthetic function families used to train and evaluate the framework, while spanning a structured and reasonably broad grid of fractional orders and convexity parameters, cannot capture

every functional form that might arise in a specific applied context, and practitioners adapting this framework to a particular application domain should expect to retrain or fine-tune the calibration layer on data representative of that domain rather than relying on the present results to transfer unconditionally. The dynamical stabilization demonstration in Figs. 8 and 9, similarly, is restricted to a single representative oscillator selected for its tractable chaotic behavior across the tested fractional order range, and broader claims about the framework's effectiveness for chaos suppression in other nonlinear fractional-order systems would require additional validation. With these limitations noted, the magnitude and consistency of the improvements documented across all six analyses support the conclusion that deliberate integration of classical fractional inequality theory with machine learning-enhanced numerical methods constitutes a productive and underexploited direction for advancing both the analytical and applied dimensions of fractional calculus research.

Conclusion

This study developed an integrated analytical-computational framework combining fractional integral inequality theory with machine learning-enhanced numerical methods for nonlinear dynamical systems and optimization. The framework incorporated four fractional operator families and three convexity classes through fractional operator embedding, neural calibration, and inequality-constrained learning.

The reported results showed an average bound-gap reduction of approximately 55%, an improvement in empirical convergence order from 2.00 to 2.16, and computational speed-up of up to ninefold at one million evaluation points. The framework also demonstrated chaos suppression in a representative fractional-order oscillator and achieved the target optimization tolerance in approximately 102 iterations. Ablation analysis identified the inequality-constrained loss and operator embedding as the most influential components.

These findings demonstrate the potential of integrating mathematical constraints with adaptive numerical learning to improve analytical sharpness, computational efficiency, and practical applicability. However, broader validation and stronger mathematical certification are required before the framework can be generalized beyond the investigated function families and application problems.

Discussion and Future Work

The findings suggest that mathematically informed machine learning can complement classical fractional inequality theory by improving bound tightness and numerical efficiency while supporting downstream applications. The strongest improvements occurred in the low fractional-order and low convexity-parameter region, indicating that adaptive sampling and region-specific model refinement may be valuable directions for future research. The ablation analysis further highlights the importance of fractional operator embedding and inequality-constrained learning, suggesting that future improvements should prioritize mathematical structure rather than relying solely on increased model complexity.

Future work should extend the framework to additional fractional operators, convexity classes, variable-order models, and more diverse function families. Stronger theoretical guarantees should also be developed to ensure that learned corrections preserve inequality validity and provide certified approximation-error bounds. Further benchmarking against optimized classical numerical methods would clarify the computational advantages of the proposed approach, particularly when training cost, inference time, memory requirements, and numerical accuracy are considered together.

The application scope should be expanded to higher-dimensional nonlinear systems, uncertain dynamical models, and more complex constrained optimization problems. Additional investigations could examine robustness under noisy observations, parameter uncertainty, and varying

initial conditions, together with the development of stability-certified control mechanisms. These extensions would help determine whether the improvements observed in the representative oscillator and optimization experiments remain consistent across broader application settings.

Independent datasets, reproducible implementations, and real-world application studies would further establish the framework's generalizability and practical reliability. Future research should also investigate transfer learning across fractional operator families and adaptive computational strategies that allocate resources according to local approximation difficulty. Collectively, these directions would support the development of a more rigorous, scalable, and application-oriented framework for fractional calculus, nonlinear dynamical systems, and optimization.

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