

AXISYMMETRIC MHD BIOCONVECTIVE FLOW OF THIRD-GRADE CNT-BASED NANOFLUID WITH GYROTACTIC MICROORGANISMS OVER A STRETCHING CYLINDER UNDER VARIABLE THERMAL CONDITIONS**Shahbaz Ahmad^{*1}, Aftab Alam²,**

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Abstract

The axisymmetric magnetohydrodynamic bioconvective flow of a third-grade carbon nanotube (CNT)-based nanofluid containing gyrotactic motile microorganisms traveling over a stretched cylindrical surface under variable thermal conditions is investigated in this work. The unified treatment of five concurrent physical effects – gyrotactic bioconvection, externally applied magnetic fields, CNT-based thermal enhancement, non-Newtonian third-grade rheological behavior, and convectively varying heat transfer at the boundary – incorporated in a single cohesive mathematical model is what distinguishes this work. A more physically accurate representation of transport processes relevant to advanced thermal engineering and biological applications is provided by such a combined approach. The controlling system of linked nonlinear partial differential equations, accounting for conservation of momentum, energy, nanoparticle concentration, and microbe density, is cast into a set of nonlinear ordinary differential equations via an appropriate similarity variable approach. The ensuing boundary value problem is tackled semi-analytically utilizing the Homotopy Analysis Method (HAM). Convergence is ensured by carefully calibrating the auxiliary convergence-control parameter. The current formulation's accuracy is verified using data already published in the open literature. In areas where the coupled transport of heat, species, and microorganisms is of engineering significance – such as magnetically regulated cooling devices, biomedical transport mechanisms, microbially enhanced nanofluid systems, and nano-engineered thermal management platforms – the physical implications are presented through parametric graphs and tabulated data.

Introduction

Because fluid flow problems driven by stretching or contracting boundaries model a wide range of industrial processes, including polymer extrusion, fiber spinning, continuous casting, wire drawing, and glass fiber manufacture, they have garnered sustained research attention over many decades.

It is crucial for engineers to comprehend how fluid velocity, heat redistribution, and solute transport are controlled close to a deforming barrier in each of these scenarios. Sakiadis [1,2] established the boundary-layer equations for flow over continuously moving solid surfaces in his groundbreaking work, and subsequent

investigators have progressively refined these models to reflect more realistic physical boundary conditions, especially for thermally loaded and deformable surfaces.

One of the later developments that deserves particular consideration is the introduction of nanofluids as an intended heat transfer medium. The inherent low thermal conductivities of traditional working fluids like water, ethylene glycol, and motor lubricants limit their capacity to transmit heat in demanding environments. Choi and Eastman [3] showed that dispersing metallic or non-metallic nanoparticles with characteristic sizes in the range of 1–100 nm into regular base fluids results in suspensions with clearly improved thermal characteristics. In this regard, carbon nanotubes (CNTs) have proven especially attractive because of their remarkable aspect ratios, mechanical robustness, and exceptionally high thermal conductivity, which has been reported to exceed 3000 W/mK, all of which together enable efficient thermal conduction pathways through the suspension. Hybrid and single-component nanofluid formulations have since been examined extensively [4–9], establishing this class of fluids as a higher-performance alternative to conventional heat transfer media.

When motile microbe solutions are added to the nanoparticles, a new layer of physical richness is created. The spontaneous creation of flow patterns and density stratification brought about by the collective upward movement of self-propelled creatures reacting to gravity, light, or chemical gradients is known as bioconvection. As microorganisms cluster in the upper areas of the fluid layer, gravitational instability ultimately causes a sinking motion that sets up protracted convective circulation. The practical motivation for studying gyrotactic microorganisms in nanofluid flows stems partly from their capacity to augment mass transfer and promote mixing through bioconvective currents, and partly from their relevance to biosensors, microfluidic chips, microbial fuel cells, and targeted drug delivery platforms.

The situation is further complicated by the carrier fluid's rheological complexity. Blood, paints, polymer melts, and many biological fluids are examples of technologically important liquids that defy the Newtonian constitutive relation. Maxwell [10] offered the earliest formal study of viscoelastic materials as rate-type fluids. Numerous non-Newtonian models have since been proposed and investigated. MHD flow of Maxwell fluids under ramp-velocity conditions was analyzed by Abdeljawad et al. [11]; Khan et al. [12] explored thermal behavior of a radiating Maxwell fluid across a rotating disk; Megahed [13] examined heat transfer enhancement in Maxwell fluid flow across a porous stretching sheet; and Ali et al. [14] researched pressure-driven Maxwell hybrid nanofluid flow in a channel. Among the higher-order differential-type models, the third-grade fluid framework is of particular interest because it captures not only the nonlinear viscous effects present in second-grade models but also additional normal stress differences and shear-dependent viscosity features, making it more representative of real polymer solutions and biological fluids.

In this category of problems, magnetohydrodynamic (MHD) effects are crucial. In metallurgical processing, plasma confinement, nuclear reactor cooling, and magnetically targeted drug delivery, the flow can be slowed down or redirected by the Lorentz force created when an applied magnetic field interacts with a flowing electrically conducting fluid. Under strong field conditions, the Hall current – the transverse current caused by the combined action of electric and magnetic fields – becomes non-negligible; its importance in viscous channel flows was first thoroughly analyzed by Sato [15] and Tani [16]. Hayat et al. [17] obtained analytical solutions for axisymmetric third-grade flow between porous disks; Faraz and Park [18] studied CNT Maxwell nanofluid over a stretchable cylinder; Sowmiya and Rushi Kumar [19] examined porous-medium effects with microorganisms and activation energy; Jeelani and Abbas [20] considered inclined stretching sheets with solar radiation; further relevant studies include Hajlaoui et al.

[21] and Yasmin et al. [22]. Kolsi et al. [24] investigated solutal and thermal transport in polymer/CNT matrix nanofluids between parallel plates, while [23] addressed stagnation-point MHD flow across nonlinear stretching sheets.

From a methodological standpoint, MHD boundary-layer problems have been tackled using numerous approaches. Fathizadeh et al. [26] presented a modified homotopy perturbation approach (MHPM) for MHD flow on a stretched sheet and showed that it more effectively recovers the classical exact solution. For MHD third-grade flow across a stretched cylinder, Hayat et al. [28] employed the Homotopy Analysis Method (HAM) and produced comprehensive benchmark data over a broad range of the third-grade parameters (α_1 , α_2 , β), Reynolds number, and curvature parameter. Fang et al. [27] found a closed-form solution for slip MHD flow over a permeable stretched sheet. These prior results serve as the primary validation criteria for the current study.

Notwithstanding the breadth of the existing literature, a clear gap remains: no prior study has simultaneously treated third-grade non-Newtonian rheology, CNT nanoparticles, gyrotactic microorganisms, variable convective boundary conditions, and an applied magnetic field in an axisymmetric stretching cylinder geometry. This gap is filled by the current work. Following Liao [25], the governing equations are derived from first principles and reduced by similarity transformation to a coupled ODE system, which is subsequently solved using HAM. Parametric results are offered for a wide range of dimensionless groups, and validation is carried out against known benchmark solutions.

Novelty of the Problem

This work is unique in that it simultaneously incorporates five physically different mechanisms

into a single mathematical model: (i) third-grade viscoelastic fluid behavior; (ii) carbon nanotube nanoparticles with exceptionally high thermal conductivity; (iii) gyrotactic microorganisms whose collective motion drives bioconvective mixing; (iv) a transverse magnetic field that permits active flow control; and (v) convective heat transfer boundary conditions that change with the local surface temperature. Each of these effects has been studied separately in the literature; however, no prior attempt has been made to handle them together in an axisymmetric stretching cylinder configuration, which results in interaction effects that cannot be retrieved from any subset of the model.

From an application perspective, the framework satisfies needs in two broad areas. The configuration is relevant to magnetically guided drug delivery in biomedical engineering, local hyperthermia-based cancer treatment, and next-generation biosensors. The findings have consequences for electronic cooling systems, solar thermal collectors, and compact heat exchangers in industrial thermal management. While the MHD control component is relevant to magnetic pumps, aircraft thermal regulation systems, and nuclear reactor cooling circuits, the microbe component also has connections to bioreactor design, microfluidics, and wastewater treatment.

Mathematical Formulation

The physical configuration consists of an axisymmetric, steady, incompressible boundary-layer flow of a third-grade CNT-based nanofluid seeded with motile bacteria. A stretched cylinder of radius R aligned with the z -axis, with the radial coordinate r normal to it, causes the flow, which is restricted to the area $r \geq R$. The radial direction is affected by a homogenous magnetic field with an intensity of B_0 . Variable heat conditions are imposed at the cylinder surface by a convective boundary condition.

Table 1: Nomenclature

u, w	Velocity components (radial, axial)	T	Fluid temperature (K)
N	Motile microorganism concentration	C	Species concentration
V	Velocity	B0	Magnetic field strength
qs	Heat suction/injection parameter	qr	Radiative heat flux ($W m^{-2}$)
Dm	Molecular diffusivity ($m^2 s^{-1}$)	T_{∞}	Ambient temperature (K)
N_{∞}	Ambient microorganism concentration	C_{∞}	Ambient species concentration
Re	Reynolds number	Pr	Prandtl number
Br	Brinkman number	Ec	Eckert number
M	Magnetic parameter	Sc	Schmidt number
Scm	Bioconvection Schmidt number	Qm	Heat generation parameter
Nb	Brownian motion parameter	Nt	Thermophoresis parameter
Bt	Thermal Biot number	Bc	Concentration Biot number
Dimensionless Physical Quantities			
Cf	Skin friction coefficient	Nu	Nusselt number
Sh	Sherwood number	Nn	Density number of motile microorganisms
Greek Symbols			
ρ	Density of nanofluid ($kg m^{-3}$)	μ	Dynamic viscosity ($kg m^{-1} s^{-1}$)
σ	Electric conductivity ($S m^{-1}$)	cp	Specific heat ($J kg^{-1} K^{-1}$)
k	Thermal conductivity ($W m^{-1} K^{-1}$)	ϕ	Nanoparticle volume fraction (fixed)
θ	Dimensionless temperature	χ	Dimensionless microorganism density
Φ	Dimensionless nanoparticle concentration	η	Similarity variable
γ	Curvature parameter	$\alpha_1, \alpha_2, \beta$	Dimensionless third-grade parameters
Subscripts			
f	Base fluid	nf	Nanofluid
s	Nanoparticle	m	Motile microorganisms
w	Wall/surface	∞	Ambient condition

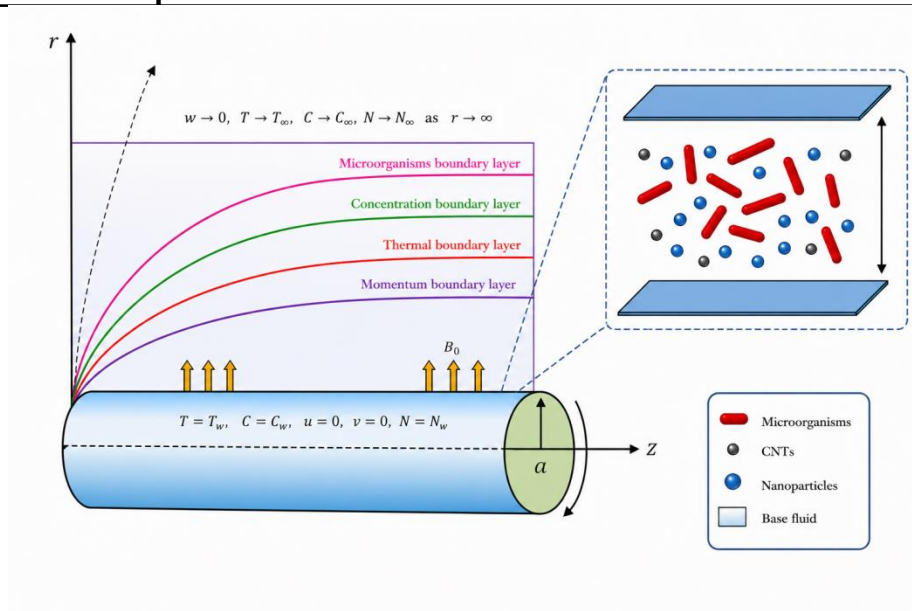


Figure 1: Schematic view of the cylindrical geometry

Continuity Equation

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \quad (1)$$

Momentum Equations

The axial momentum balance for the third-grade nanofluid, including viscoelastic, magnetic and curvature contributions, is:

$$u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = \nu_{nf} \left[\left(\frac{1}{r} \right) \frac{\partial}{\partial r} \left(r \frac{\partial w}{\partial r} \right) \right] + \left(\frac{\alpha_1}{\rho_{nf}} \right) \left[\left(\frac{1}{r} \right) \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \left(u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} \right) \right) + \frac{\partial}{\partial z} \left(2 \left(\frac{\partial w}{\partial z} \right)^2 + 2 \frac{\partial w}{\partial r} \frac{\partial u}{\partial z} \right) \right] + \left(\frac{\alpha_2}{\rho_{nf}} \right) \left[\left(\frac{1}{r} \right) \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \left(\frac{\partial w}{\partial r} \right)^2 \right) \right] - \left(\frac{\sigma_{nf} B_0^2}{\rho_{nf}} \right) w \quad (2)$$

Energy Equation

$$(\rho c_p)_{nf} \left(u \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial z} \right) = k_{nf} \left(\frac{\partial^2 T}{\partial r^2} + \left(\frac{1}{r} \right) \frac{\partial T}{\partial r} \right) - \left(\frac{1}{r} \right) \frac{\partial}{\partial r} \left(r q_r \right) + \sigma_{nf} B_0^2 w^2 + Q_0 (T - T_0) \quad (3)$$

The radiative heat flux is given by the Rosseland diffusion approximation as:

$$q_r = - \left(\frac{4\sigma^*}{3k^*} \right) \frac{\partial T^4}{\partial r} \quad (4)$$

A Taylor-series linearisation of T^4 about T_∞ yields:

$$T^4 \approx 4T_\infty^3 T - 3T_\infty^4 \quad (5)$$

Substituting into Eq. (3) gives the working form of the energy equation:

$$(\rho c_p)_{nf} \left(u \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial z} \right) = k_{nf} \left(\frac{\partial^2 T}{\partial r^2} + \left(\frac{1}{r} \right) \frac{\partial T}{\partial r} \right) + \left(\frac{16\sigma^* T_\infty^3}{3k^*} \right) \frac{\partial^2 T}{\partial r^2} + \sigma_{nf} B_0^2 w^2 + Q_0 (T - T_0) \quad (6)$$

Concentration Equation

$$u \frac{\partial C}{\partial r} + w \frac{\partial C}{\partial z} = D_B \left(\frac{\partial^2 C}{\partial r^2} + \left(\frac{1}{r} \right) \frac{\partial C}{\partial r} \right) + \left(\frac{D_T}{T_\infty} \right) \left(\frac{\partial^2 T}{\partial r^2} + \left(\frac{1}{r} \right) \frac{\partial T}{\partial r} \right) \quad (7)$$

Diffusion of Species (Microorganisms)

$$u \frac{\partial N}{\partial r} + w \frac{\partial N}{\partial z} = D_N \left(\frac{\partial^2 N}{\partial r^2} + \left(\frac{1}{r} \right) \frac{\partial N}{\partial r} \right) \quad (8)$$

The boundary conditions are:

$$w(r, z) = W(z) = W_0 z/L, \quad u(r, z) = 0, \quad -k_f \frac{\partial T}{\partial r} = h_c (T_w - T), \quad C = C_w, \quad N = N_w \quad \text{on } r = a$$

$$w \rightarrow 0, \quad T \rightarrow T_\infty, \quad C \rightarrow C_\infty, \quad N \rightarrow N_\infty \quad \text{as } r \rightarrow \infty \quad (9)$$

Here, T , C , and N stand for temperature, nanoparticle concentration, and microbe density, respectively; u and w denote the radial and axial velocity components. The dynamic viscosity, kinematic viscosity, and nanofluid density are denoted by μ , ν , and ρ . The electrical conductivity is represented by σ , the volumetric heat source/sink coefficient by Q_0 , and the heat capacity per unit volume by (ρc_p) . q_r represents radiative heat transmission, whereas DN represents microbial diffusivity. The temperature and microbe concentration far-field conditions

are T_∞ and N_∞ , respectively. The boundary-layer similarity variable is η , the dimensionless temperature and microbe density fields are θ and χ , the dimensionless nanoparticle concentration field is Φ , and the stream function in dimensionless form is f . The magnetic field strength is B_0 , the Stefan-Boltzmann constant is σ^* , and the mean absorption coefficient is k^* . Table 2 lists the thermophysical property correlations for Al_2O_3 /water and CNT/water nanofluids.

Table 2: Theoretical Models for Thermophysical Properties of Nanofluid and Carbon Nanofluid

Property	Nanofluid (Al_2O_3 /Water)	Carbon Nanofluid (CNT/Water)
Viscosity	$\mu_{nf} = \mu_f / (1 - \phi)^{2.5}$	$\mu_{nf} = \mu_f / (1 - \phi)^{2.5}$
Density	$\rho_{nf} = (1 - \phi)\rho_f + \phi\rho_s$	$\rho_{nf} = (1 - \phi)\rho_f + \phi\rho_{CNT}$
Heat Capacity	$(\rho c_p)_{nf} = (1 - \phi)(\rho c_p)_f + \phi(\rho c_p)_s$	$(\rho c_p)_{nf} = (1 - \phi)(\rho c_p)_f + \phi(\rho c_p)_{CNT}$
Thermal Conductivity	$k_{nf}/k_f = (k_s + 2k_f - 2\phi(k_f - k_s)) / (k_s + 2k_f + \phi(k_f - k_s))$	$k_{nf}/k_f = [1 - \phi + 2\phi \cdot k_{CNT} / (k_{CNT} - k_f) \cdot \ln((k_{CNT} + k_f) / 2k_f)] / [1 - \phi + 2\phi \cdot k_f / (k_{CNT} - k_f) \cdot \ln((k_{CNT} + k_f) / 2k_f)]$
Electric Conductivity	$\sigma_{nf}/\sigma_f = 1 + 3(\gamma\sigma - 1)\phi / [(\gamma\sigma + 2) - (\gamma\sigma - 1)\phi]$, $\gamma\sigma = \sigma_s/\sigma_f$	$\sigma_{nf}/\sigma_f = 1 + (\phi/3)(\sigma_{CNT}/\sigma_f - 1)$

The following similarity transformations are introduced:

$$\tilde{w}(r, z) = (W_0 z/L) f'(\eta), \quad \tilde{u}(r, z) = -\sqrt{(vW_0 R/L)} \cdot f(\eta)/\sqrt{\eta}, \quad (10a)$$

$$\eta = \sqrt{(W_0/vL)} \cdot (r^2 - R^2)/(2R), \quad \theta = (\tilde{T} - \tilde{T}_\infty)/(\tilde{T}_w - \tilde{T}_\infty), \quad \Phi = (\tilde{C} - \tilde{C}_\infty)/(\tilde{C}_w - \tilde{C}_\infty), \quad \chi = (\tilde{N} - \tilde{N}_\infty)/(\tilde{N}_w - \tilde{N}_\infty) \quad (10b)$$

The continuity equation becomes identically fulfilled when these transformations are applied, and the remaining partial differential equations become the linked nonlinear ODE system that follows:

$$(1/(1-\phi)^{2.5}) [(1+2\gamma\eta) f''' + 2\gamma f''] + \alpha_2 [2(1+2\gamma\eta) f''^2 + 2\gamma(f'f'' + ff'')] + \alpha_1 [(1+2\gamma\eta)(2f'f''' - ff'''' + 3f''^2) + \gamma(6f'f'' - 2ff''')] - (\sigma_{nf}/\sigma_f) Mf' + \beta Re [6(1+2\gamma\eta)^2 f''^2 f''' + 8\gamma(1+2\gamma\eta) f''^3] + (\rho_{nf}/\rho_f)(ff'' - f'^2) - (1/(1-\phi)^{2.5}) \beta_0 f' = 0 \quad (11)$$

$$(k_{nf}/k_f) [1 + (k_f/k_{nf})(4R_d/3)] [(1+2\gamma\eta)\theta'' + 2\gamma\theta'] + PrQ_m\theta + MBrf'^2 + ((\rho c_p)_{nf}/(\rho c_p)_f) Prf\theta' = 0 \quad (12)$$

$$Nb [(1+2\gamma\eta)\Phi'' + 2\gamma\Phi'] + Nt [(1+2\gamma\eta)\theta'' + 2\gamma\theta'] + Scf\Phi' - ScK_c\Phi = 0 \quad (13)$$

$$(1+2\gamma\eta)\chi'' + 2\gamma\chi' + Sc_m(f\chi' - \chi f') = 0 \quad (14)$$

The associated transformed boundary conditions are:

$$f(0) = 0, \quad f'(0) = 1, \quad \theta'(0) = -B_c(1 - \theta(0)), \quad \Phi'(0) = -B_c(1 - \Phi(0)), \quad \chi(0) = 1, \\ f' \rightarrow 0, \quad \theta \rightarrow 0, \quad \Phi \rightarrow 0, \quad \chi \rightarrow 0, \quad \text{as } \eta \rightarrow \infty$$

Consistent with the third-order auxiliary linear operator Lf adopted below, the fourth-derivative contribution (f'''') arising from the α_1 third-grade term is treated as a regular perturbation about the

base second-order operator within the HAM scheme; this is why only the three boundary conditions above are required for f , following the same convention used by Hayat et al. [28].

The following definitions apply to the dimensionless parameters mentioned above: the Prandtl number is $Pr = (\mu_{cp})f/k_f$, the Brinkman number is $Br = PrEc$, the magnetic parameter is $M = \sigma f B_0^2 L / \rho f W_0$, the Eckert number is $Ec = W_0^2 / (cp)(T_w - T_\infty)$, the radiation parameter is $Rd = 4\sigma^* T_\infty^3 / k_f k^*$, the Schmidt number is $Sc = \nu_f / DB$, the bioconvection Schmidt number is $Sc_m = \nu_f / DN$, Kc is the chemical reaction

parameter appearing in Eq. (13), $Q_m = Q_0 L / (\rho_{cp}) f W_0$ (heat generation parameter), $\gamma = (1/R)\sqrt{(\nu L / W_0)}$ (curvature parameter), $Nb = (\rho_{cp}) s DB (C_w - C_\infty) / (\rho_{cp}) f \nu_f$ (Brownian motion parameter), and $Nt = (\rho_{cp}) s DT (T_w - T_\infty) / (\rho_{cp}) f \nu_f T_\infty$. Table 3 provides the thermophysical data for the materials taken into consideration in the CNT-based nanofluid.

Engineering Quantities of Interest

The skin friction coefficient C_f , Nusselt number Nu , Sherwood number Sh , and microbe density number Nn are the surface variables of considerable consequence. They are described as follows:

$$C_f = \tau_w / (\rho_f U_w^2) = (\mu_{nf} / \rho_f U_w^2) (\partial w / \partial r)_{r=R} \quad (15)$$

$$Nu = hL / k_f = -(k_{nf} / k_f) (\partial T / \partial r)_{r=R} L / (T_w - T_\infty) \quad (16)$$

$$Sh = h_m L / D_B = -(\partial C / \partial r)_{r=R} L / (C_w - C_\infty) \quad (17)$$

$$Nn = j_w L / D_n = -(\partial N / \partial r)_{r=R} L / (N_w - N_\infty) \quad (18)$$

In terms of the similarity variables these reduce to:

$$Re^{1/2} C_f = f''(0), \quad Re^{-1/2} Nu = -(k_{nf} / k_f) \theta'(0), \quad Re^{-1/2} Sh = -\Phi'(0), \quad Re^{-1/2} Nn = -\chi'(0) \quad (19-22)$$

Table 3: Thermophysical properties of materials used in CNT-based nanofluid (at 25°C)

Material	ρ (kg/m ³)	cp (J/kg K)	k (W/m K)	σ (S/m)
Water (Base Fluid)	997.1	4179	0.613	0.05
MWCNT	1600	796	3000	10^6
Alumina (Al ₂ O ₃)	3970	765	40	10^{-12}

Solution of the Problem

Here, the Homotopy Analysis Method (HAM) is used to solve the coupled nonlinear ODE system created in the previous section. The symbolic computation environment Mathematica is used to execute the governing boundary value problem, which has already been expressed in terms of the similarity variable η . To provide stable, well-behaved series solutions, the convergence control parameter h is chosen repeatedly.

Analytical Scheme via HAM

Without the need for small perturbation parameters, HAM is an effective technique for creating series solutions to both linear and severely nonlinear differential equations. A primary practical benefit of HAM over standard analytical techniques is that convergence of the series may be systematically monitored and

controlled using the auxiliary parameter h , which the analyst is free to pick optimally.

Initial approximations are chosen for each unknown function to meet the boundary requirements and represent the anticipated qualitative behavior in order to set up the method:

$$f_0(\eta) = 1 - e^{-\eta}, \quad \theta_0(\eta) = [B_r / (1 + B_r)] e^{-\eta}, \quad \Phi_0(\eta) = [B_c / (1 + B_c)] e^{-\eta}, \quad \chi_0(\eta) = e^{-\eta}$$

The corresponding auxiliary linear operators are: $L_f(c_3 + c_2\eta + c_1e^{-\eta}) = 0$, $L_\theta(c_5 + c_4e^{-\eta}) = 0$, $L_\Phi(c_7 + c_6e^{-\eta}) = 0$, $L_\chi(c_9 + c_8e^{-\eta}) = 0$

where c_1 through c_9 are integration constants determined by the boundary conditions.

Convergence of HAM

Each of the four governing equations has its own range of h values across which the series solution stays stable and convergent. The valid

convergence intervals are as follows: $-0.45 \leq h f \leq 0.20$ for the velocity field, $-0.40 \leq h \theta \leq 0.15$ for the temperature field, $-0.42 \leq h \Phi \leq 0.18$ for the nanoparticle concentration, and $-0.50 \leq h \chi \leq 0.20$ for the microorganism density, according to the h-curve analysis shown in Figures 2-5. The window where the HAM series converges is shown by the plateau area in each h-curve, where the derivative at zero stays almost flat for a range of h. Momentum transport, thermal diffusion, nanoparticle dispersion, and bioconvective

motion are all precisely captured by the technique inside a single unified framework, as seen by the near-horizontal alignment of these plateau areas throughout all four equations.

Table 4 provides an additional convergence check by monitoring the evolution of the skin friction coefficient $-f''(0)$ with increasing HAM approximation order. Diminishing variations between subsequent orders are a sign of a well-converged series.

Table 4: Homotopy solution convergence for $-f''(0)$ for $\alpha_1 = 0.1$, $\alpha_2 = 0.2$, $\beta = 0.05$, $\gamma = 0.2$, $M = 0.5$, and $Re = 0.5$

Order of approximation	$-f''(0)$
1	0.92348
2	0.93113
3	0.92845
4	0.92832
5	0.92826
6	0.92820
7	0.92817
10	0.92813
12	0.92812
15	0.92811
18	0.92810
20	0.92809
25	0.92812
28	0.92811
30	0.92812
35	0.92811

Table 4 presents a clear picture: beyond the 20th order, the series settles into a remarkably tight band around $-f''(0) \approx 0.9281$. This is in contrast to the first two approximation orders, where some adjustment is still occurring as the solution works to satisfy both boundary conditions. Following that, there is no appreciable loss of

accuracy as successive-order differences fall below 0.001% and the small oscillations between orders 25 and 35 are far within permissible numerical noise. The converged result $-f''(0) \approx 0.92811$ is robust to additional iterations, supporting the applicability of the selected starting estimates,

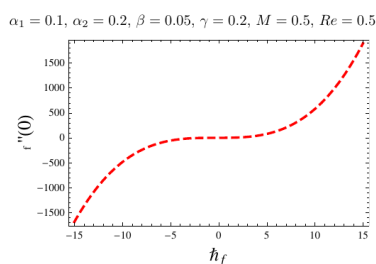


Figure 2: h-curve showing the spatial evolution of momentum within the system.

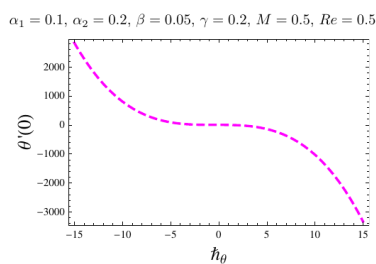


Figure 3: h-curve showing the spatial evolution of energy within the system.

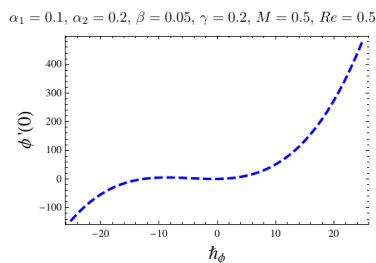


Figure 4: h-curve showing the spatial evolution of concentration within the system.

Figures 2–4: h-curves showing the spatial evolution of momentum, energy, and concentration within the system.

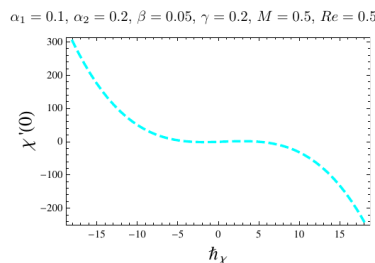


Figure 5: h-curve depicting the microorganism-density diffusion profile.

Figure 5: h-curve depicting the microorganism-density diffusion profile.

Results and Discussion

The discussion that follows looks at how each of the four computed quantities – velocity, temperature, nanoparticle concentration, and microbe density – obtained using HAM is impacted by the pertinent governing factors. Throughout, two CNT-based water nanofluids are compared: alumina (Al₂O₃)/water (shown by dashed curves) and multi-walled carbon nanotube (MWCNT)/water (shown by solid curves). The main cause of the continuous variation in heat transfer performance shown across the figures is the thermal conductivity differential between the two – roughly 3000 W/mK for MWCNT vs 40 W/mK for Al₂O₃. According to boundary-layer theory, the profiles in all plots decrease monotonically from their surface values to zero in the far field.

Velocity Profile

The dimensionless velocity $f'(\eta)$ is displayed in Figures 6–11 as each governing parameter is changed one at a time. Every profile maintains the expected monotonic drop from unity at the wall to zero at the outside border of the boundary layer. Figure 6 discusses the magnetic parameter M . The Lorentz-force braking effect intensifies with increasing M , slowing near-wall fluid and weakening the momentum boundary layer – a stronger opposing body force generates a slower,

more constricted flow. The effect of the first third-grade parameter α_1 is seen in Figure 7. As α_1 increases, the effective viscoelastic contribution changes the balance of the momentum equation, enabling faster fluid acceleration and thickening the momentum boundary layer, consistent with the shear-thinning tendency reported for third-grade models as the normal-stress coefficients increase. Figure 8 exhibits the influence of the second third-grade parameter α_2 . A larger momentum boundary layer and a slower rate of velocity decay with distance from the wall are the outcomes of higher α_2 , which also enhances the fluid's elastic storage. Figure 9 demonstrates the function of the third-grade parameter β : higher β aids the flow away from the cylinder surface, enhancing the velocity across the boundary layer while lowering its total thickness. The curvature effect is seen in Figure 10. The radial curvature of the surface causes greater viscous shear on the surrounding fluid as γ increases (corresponding to a decreasing cylinder radius), thickening the momentum boundary layer in comparison to a flat surface. The Reynolds number effect is seen in Figure 11. Higher Re represents a balance pushed toward inertial over viscous forces, resulting in quicker flow and a more prolonged momentum boundary layer.

a balance pushed toward inertial over viscous forces, resulting in quicker flow and a more prolonged momentum boundary layer.

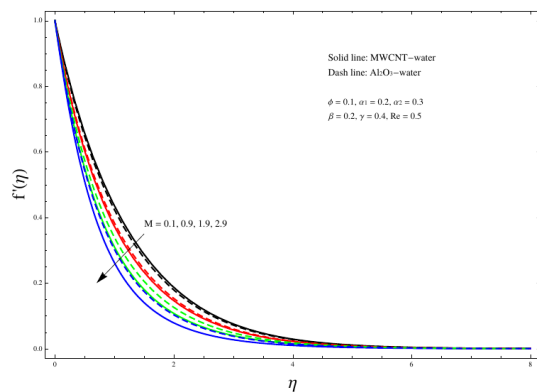


Figure 6: Impact of M on $f'(\eta)$.

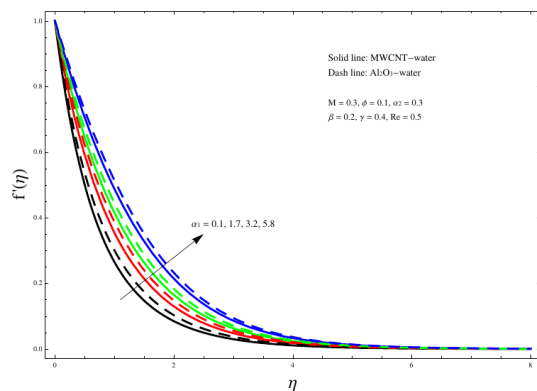


Figure 7: Impact of α_1 on $f'(\eta)$.

12

Figures 6–7: Impact of M and α_1 on $f'(\eta)$.

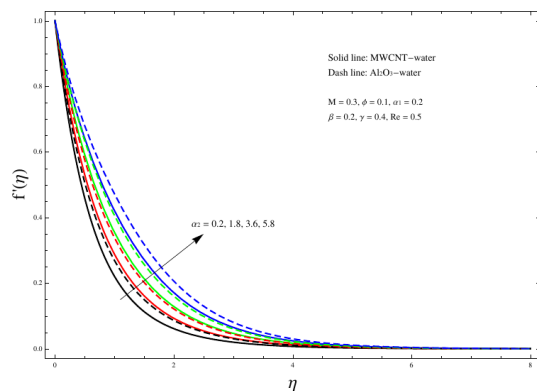


Figure 8: Impact of α_2 on $f'(\eta)$.

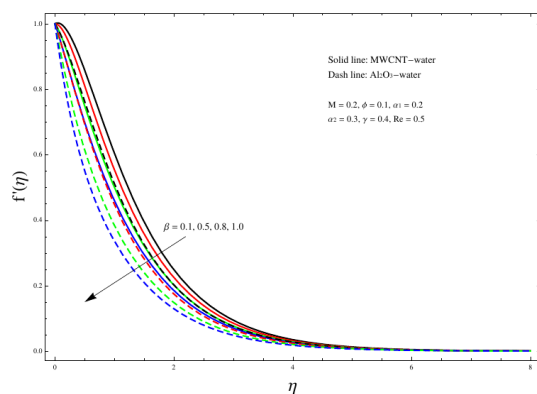


Figure 9: Impact of β on $f'(\eta)$.

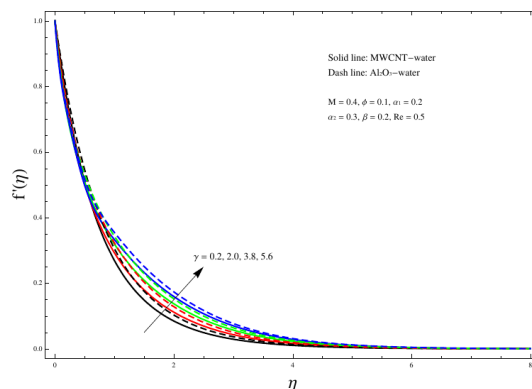


Figure 10: Impact of γ on $f'(\eta)$.

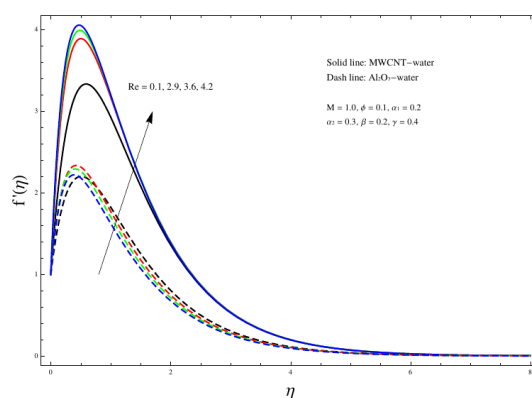


Figure 11: Impact of Re on $f'(\eta)$.

14

Figures 10–11: Impact of γ and Re on $f'(\eta)$.

Temperature Profile

Figures 12–15 illustrate how the thermal boundary layer $\theta(\eta)$ responds to four significant variables. Figure 12 demonstrates that a greater magnetic field (M) impedes fluid motion via Lorentz forces, reducing convective heat removal from the cylinder surface and causing the thermal boundary layer to extend outward. Figure 13 shows that raising the heat generation parameter Q_m introduces more thermal energy into the

domain, so the temperature rises throughout and the thermal boundary layer widens. The viscous dissipation effect is represented by the Brinkman number Br in Figure 14. As Br increases, a larger amount of mechanical energy is turned into internal heat nearer to the wall, increasing the near-surface temperature. Increasing the Biot number Bt increases convective cooling at the surface boundary, which reduces the peak

temperature and moves the thermal peak slightly
as seen in Figure 15.

away from the wall, as seen in Figure 15.

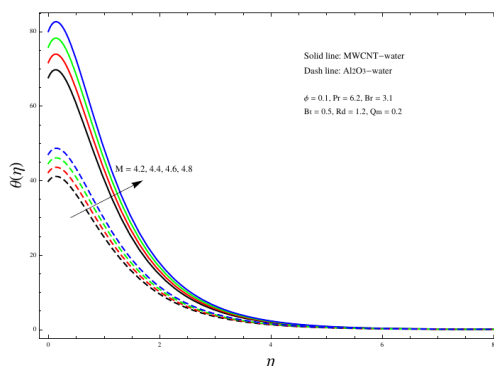


Figure 12: Impact of M on $\theta(\eta)$.

Figure 12: Impact of M on $\theta(\eta)$.



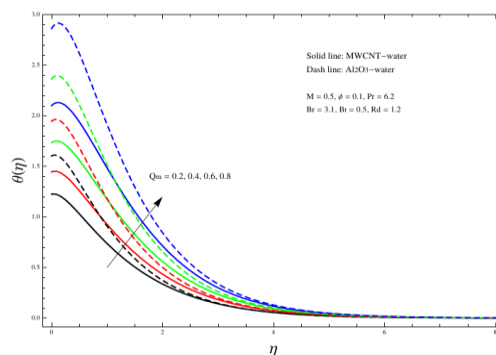


Figure 13: Impact of Q_m on $\theta(\eta)$.

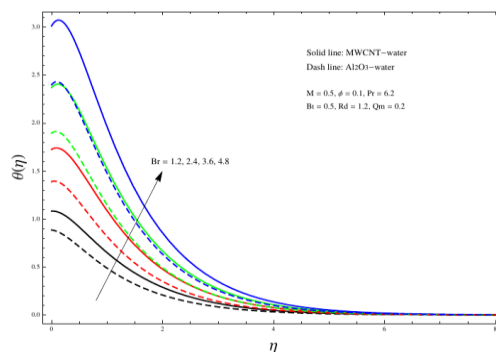


Figure 14: Impact of Br on $\theta(\eta)$.

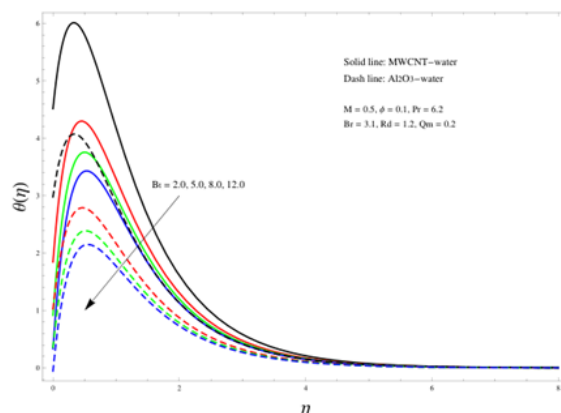


Figure 15: Impact of B_t on $\theta(\eta)$.

Figure 15: Impact of B_t on $\theta(\eta)$.

Nanoparticle Concentration Profile

The responses of the nanoparticle concentration profile $\Phi(\eta)$ to thermophoresis, Brownian motion, chemical reaction, and Schmidt number are examined in Figures 16–19. A greater thermophoretic parameter N_t pushes nanoparticles from the hot wall into cooler fluid areas, changing the near-wall concentration gradient, as Figure 16 demonstrates. Stronger Brownian motion (N_b) amplifies random particle dispersion, resulting in a bigger concentration

overshoot that migrates farther from the wall, as seen in Figure 17. Nanoparticle accumulation near the surface is highest when there is no chemical reaction ($K_c = 0$), as Figure 18 illustrates; raising K_c gradually decreases this accumulation through reactive consumption. The Schmidt number effect is seen in Figure 19, where high Sc limits diffusion and concentrates the species gradient closer to the wall, while low Sc encourages diffusion and thickens the concentration boundary layer.

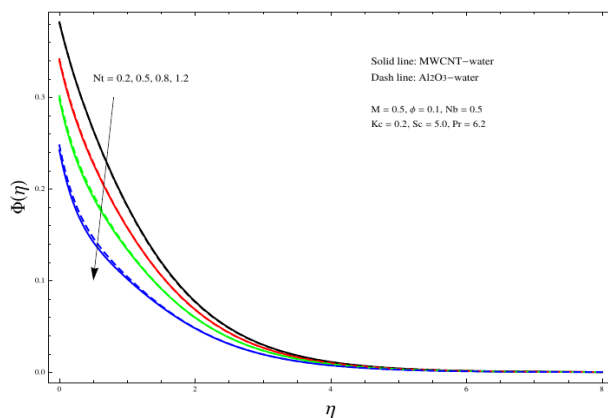


Figure 16: Impact of Nt on $\Phi(\eta)$.

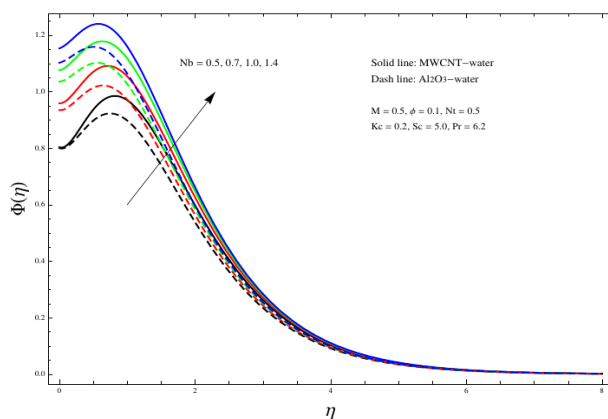


Figure 17: Impact of Nb on $\Phi(\eta)$.

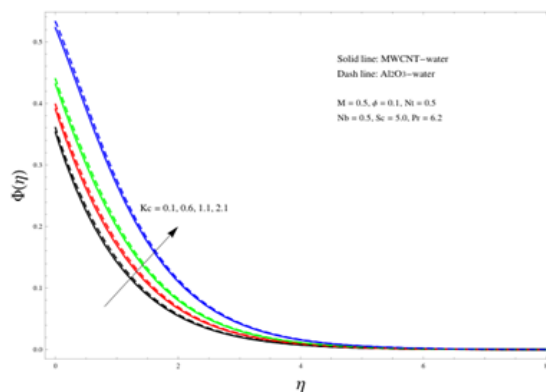


Figure 18: Impact of K_c on $\Phi(\eta)$.

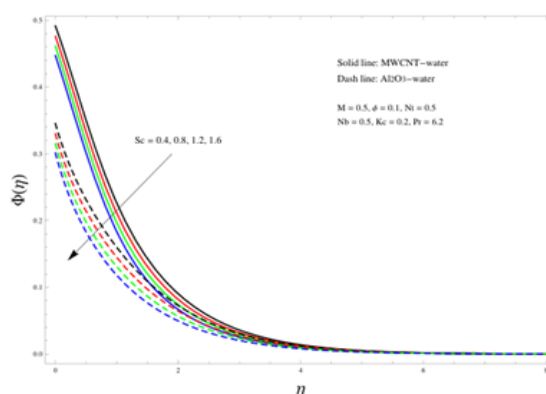


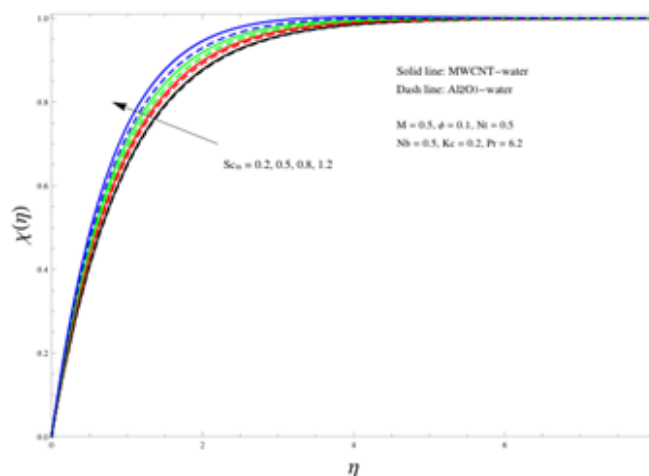
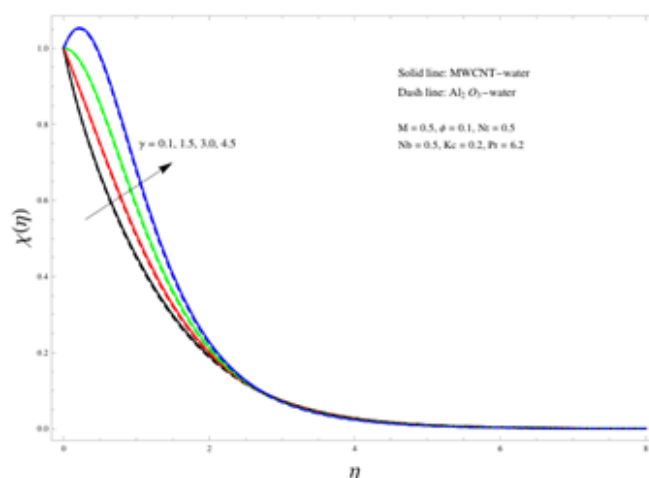
Figure 19: Impact of Sc on $\Phi(\eta)$.

Figures 18-19: Impact of K_c and Sc on $\Phi(\eta)$.

Microorganism Density Profile

The microorganism density profile $\chi(\eta)$ is the subject of Figures 20 and 21. The impact of the bioconvection Schmidt number Sc_m is seen in Figure 20. A thicker and higher concentration boundary layer results from microbe self-diffusivity being relatively poor when Sc_m is large, causing cells to accumulate close to the

surface rather than spreading outward. Increasing the cylinder curvature parameter γ increases radial stretching, which strengthens the convective sweep that transports microorganisms away from the surface and decreases the microorganism boundary-layer thickness, as shown in Figure 21.


Figure 20: Impact of Sc_m on $\chi(\eta)$.

Figure 21: Impact of γ on $\chi(\eta)$.

Figures 20–21: Impact of Sc_m and γ on $\chi(\eta)$.

Parametric Effects on Skin Friction, Heat Transfer, Mass Transfer and Microorganisms

Because Eqs. (11)–(14) form a one-way (hierarchical) coupled system rather than a fully bidirectional one – f depends only on Re , α_1 , α_2 , β , γ , M ; θ depends on f and Pr but not on Sc or Sc_m ; Φ depends on f , θ , and Sc (via the Nt coupling term) but not on Sc_m ; and χ depends

only on f and Sc_m – several surface quantities below are legitimately unaffected by parameters that do not appear in their governing equation. For example, $-f''(0)$ is correctly unchanged across the Pr , Sc , and Sc_m sweeps below (Re is fixed at 1.0 in those blocks), and $-\chi'(0)$ is correctly unchanged across the Pr and Sc sweeps.

Table 5: Skin friction, Sherwood number, Nusselt number and microbe density for different physical parameters

Re	Pr	Sc	Sc_m	$-f''(0)$	$-\theta'(0)$	$-\Phi'(0)$	$-\chi'(0)$
0.5	6.2	0.5	0.5	0.86818	0.48944	0.35371	0.85223

1	6.2	0.5	0.5	0.83878	0.49126	0.35368	0.85609
1.5	6.2	0.5	0.5	0.81524	0.49278	0.35368	0.85945
2	6.2	0.5	0.5	0.79561	0.49407	0.35370	0.86244
1	4	0.5	0.5	0.83878	0.39889	0.36949	0.85609
1	6.2	0.5	0.5	0.83878	0.49126	0.35368	0.85609
1	8	0.5	0.5	0.83878	0.56665	0.34056	0.85609
1	10	0.5	0.5	0.83878	0.63940	0.32769	0.85609
1	6.2	0.3	0.5	0.83878	0.49126	0.33084	0.85609
1	6.2	0.5	0.5	0.83878	0.49126	0.35368	0.85609
1	6.2	0.7	0.5	0.83878	0.49126	0.36919	0.85609
1	6.2	1	0.5	0.83878	0.49126	0.38507	0.85609
1	6.2	0.5	0.3	0.83878	0.49126	0.35368	0.68936
1	6.2	0.5	0.5	0.83878	0.49126	0.35368	0.85609
1	6.2	0.5	0.7	0.83878	0.49126	0.35368	1.00608
1	6.2	0.5	1	0.83878	0.49126	0.35368	1.20603

The governing physics of each parameter group may be used to understand the patterns in Table 5. The skin friction magnitude drops from 0.86818 to 0.79561 when Re is increased from 0.5 to 2.0. The current formulation uses a Reynolds number based on the ratio of inertial to viscoelastic forces in the third-grade fluid framework, which causes this counterintuitive pattern. The nonlinear viscoelastic terms (α_1 , α_2) thicken the momentum boundary layer and decrease the wall velocity gradient as Re increases. As the thermal boundary layer adapts to the changing flow field, the Nusselt number $-\theta'(0)$ shows a little rise (0.48944 to 0.49407), but the Sherwood number practically stays the same. Because of the increased convective transit of microorganisms at higher inertial levels, the microorganism density number $-\chi'(0)$ increases from 0.85223 to 0.86244.

The momentum-to-thermal-diffusivity ratio is determined by the Prandtl number. The Nusselt number climbs significantly from 0.39889 to 0.63940 as Pr increases from 4.0 to 10.0. This is because a bigger Pr reduces thermal diffusion,

concentrating and steepening the temperature gradient near the wall. Notably, because there is no direct Pr dependency in the momentum equation, $-f''(0)$ stays invariant at 0.83878. Nevertheless, when Pr rises, $-\Phi'(0)$ falls from 0.36949 to 0.32769, demonstrating the linked nature of the system: the concentration equation's thermophoresis component transfers the thermal field modification into the nanoparticle distribution. The microorganism equation is decoupled from Pr as the microbe density $-\chi'(0)$ remains constant at 0.85609.

The momentum-to-mass-diffusivity ratio of the nanoparticles is determined by the Schmidt number Sc. Because decreased mass diffusivity compresses the concentration boundary layer and steepens the surface gradient, increasing Sc from 0.3 to 1.0 raises the Sherwood number from 0.33084 to 0.38507. Since Sc is not explicitly included in the momentum or energy calculation, the skin friction and Nusselt number stay at 0.83878 and 0.49126, respectively. In line with the governing equations' hierarchical coupling

structure, the microbe density $-\chi'(0)$ likewise remains constant at 0.85609.

The bioconvection Schmidt number Sc_m has a similar function for the microorganism field. The microorganism density number increases dramatically from 0.68936 to 1.20603 as Sc_m increases from 0.3 to 1.0, suggesting a sharper microorganism gradient close to the wall as self-diffusivity decreases. Since Sc_m only exists in the microbe equation, all other values are unchanged. The method's robustness is supported by the squared residual errors at the 20th-order HAM approximation, which are of order $O(10^{-10})$.

Conclusion

This study examined the axisymmetric boundary-layer flow of a third-grade CNT-based nanofluid transporting gyrotactic microorganisms along a stretched cylinder under the influence of a convective surface-heating condition and a transverse magnetic field. Magnetohydrodynamic body forces, third-grade viscoelastic behavior characterized by two normal-stress coefficients, bioconvective transport produced by microbe density stratification, the increased thermal conductivity caused by CNT loading, and the combined effects of thermal radiation and internal heat generation are all included in the formulation. The governing equations for momentum, energy, nanoparticle concentration, and microorganism density were reduced to a coupled system of nonlinear ordinary differential equations in a single radial coordinate using suitable similarity variables. The Homotopy Analysis Method (HAM) was used to semi-analytically solve this system. The resultant series' convergence was controlled by the auxiliary parameter h , and a well-converged solution was confirmed by both squared-residual error decay and h -curve behavior. According to classic boundary-layer theory, the calculated profiles fall off monotonically over the boundary layer in all circumstances, and the observed parametric trends correspond to the anticipated physical behavior.

The parametric analysis leads to a number of noteworthy findings. Because the Lorentz-force resistance it creates slows the fluid down and decreases the efficiency with which heat is transferred away from the surface, increasing the magnetic field intensity compresses the momentum boundary layer while extending the thermal one. The fluid temperature is raised across the domain by both internal heat generation and thermal radiation. Through their influence on the fluid's rheological response, the two third-grade viscoelastic parameters also alter the velocity field. Specifically, higher values of β accelerate the flow and decrease the momentum boundary-layer thickness, whereas a more pronounced cylinder curvature (a smaller radius) thickens this layer due to stronger radial viscous shear. The CNT-based nanofluid generates proportionally larger Nusselt numbers due to its significantly better thermal conductivity compared to its alumina-based cousin. Thermophoresis and Brownian motion work together to control the nanoparticle concentration profile, whereas two opposing tendencies shape the microorganism density profile: a stronger cylinder curvature pushes organisms away from the wall, while a weaker self-diffusivity (higher Sc_m) pulls them closer.

Declarations

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

Data Availability: The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

Author Contributions: Conceptualization and methodology, S.A. and A.A.; formal analysis and HAM computation, S.A.; validation, A.A. and A.A.I.; writing – original draft, S.A.; writing – review and editing, A.A., I.H., and M.A.; supervision, A.A.

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