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THE JK ITERATIVE SCHEME FOR GENERALIZED α-NONEXPANSIVE MAPPINGS: CONVERGENCE THEORY, A NEW NON-SUZUKI EXAMPLE, AND A COST-ADJUSTED NUMERICAL COMPARISON

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Abstract

The JK (Junaid–Kifayat) iteration is a three-step scheme reported to converge faster than several classical schemes when approximating fixed points of Suzuki and generalized α-nonexpansive mappings. We give a self-contained treatment of its convergence theory for generalized α-nonexpansive mappings in uniformly convex Banach spaces. The proofs rest on the inequality $\|x - Ty\| \le \frac{3+\alpha}{1-\alpha}\|x - Tx\| + \|x - y\|$ *and they show that only the first averaging parameter has to be bounded away from 0 and 1. We then construct a new piecewise-affine self-map of [0, 4] and prove that it is generalized α-nonexpansive with the optimal value α = 1/5, while being neither nonexpansive nor a Suzuki mapping. On this map we compare JK with the Mann, Ishikawa, Noor, Agarwal, Abbas–Nazir and Thakur schemes for six parameter settings, five starting points and two tolerances, using 50-digit arithmetic. For starting points in the affine part the error is multiplied by an exactly computable factor at every iteration; we prove that JK's factor is strictly the smallest when all parameters coincide, and the predicted iteration counts agree with the observed ones in every tested case. JK needs the fewest iterations in all 60 experiments. However, it uses four evaluations of the map per iteration, and Thakur's scheme needs fewer evaluations in 47 of the 60 experiments; JK's advantage in cost appears only for parameters above roughly 0.6.*

INTRODUCTION

Many problems in analysis can be written as a fixed point problem $Tu = u$ for a self-map T of a set M in a Banach space X. Banach's contraction principle [5] gives existence, uniqueness and convergence of Picard iteration for contractions. Nonexpansive mappings, $\|Tu - Tv\| \le \|u - v\|$, are the natural next class, but for them Picard iteration may fail to converge even when a fixed point exists. Existence is guaranteed on bounded closed convex subsets of uniformly convex spaces [6, 9, 11], and this

has led to a long line of iterative schemes: Mann [12], Ishikawa [10], Noor [13], Agarwal–O'Regan–Sahu [2], Abbas–Nazir [1], Thakur–Thakur–Postolache [21] and the M iteration of Ullah and Arshad [23].

Suzuki [20] enlarged the class of nonexpansive mappings to mappings satisfying condition (C), and Pant and Shukla [17] enlarged it further to the *generalized* α*-nonexpansive* mappings. The JK scheme was introduced by Ahmad, Ullah, Arshad and Ma [4] for Suzuki mappings and extended

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to generalized α-nonexpansive mappings by Ahmad, I¸sık, Ali, Ullah, Ameer and Arshad [3]; Ullah et al. [22] treated mappings with condition (E) and proved stability and data-dependence results. The published numerical evidence for JK consists of comparisons on one or two examples and a few parameter choices. 2

Contributions.

This paper has four parts.

Theory (Section 3). We give complete proofs of weak 2 and strong convergence of JK for generalized α-nonexpansive mappings, working through the inequality of Lemma 2.4(c). The proofs make clear which hypotheses are used: only $\xi_k \in [a, b] \subset (0, 1)$ for the first parameter is needed, while $\varpi_k \in [0, 1]$ is arbitrary. For contractions we obtain a parameter-free rate δ^2 per iteration. 2

A new example (Section 4). We construct a map on [0, 4] with a jump, prove it is generalized α-nonexpansive with $\alpha = 1/5$ (and that no smaller α works), and exhibit explicit points at which nonexpansiveness and Suzuki's condition (C) fail.

Exact rates (Section 5.2). On the affine part of the map every scheme multiplies the error by an explicit constant per step. We prove that JK has the strictly smallest constant when all parameters coincide, and that the constants predict the observed iteration counts exactly.

A cost-adjusted comparison (Section 5.3). We report iterations *and* evaluations of T , since JK evaluates T four times per iteration, whereas Thakur's scheme uses three. The picture changes noticeably once cost is counted.

The convergence theorems are, in substance, those of [3, 4]; our contribution there is a complete and streamlined proof. The example, the exact-rate analysis and the cost-adjusted study are new to this paper as far as we know; we do not claim priority for the general type of example, which is in the spirit of those of Suzuki [20] and Pant–Shukla [17].

Preliminaries

Throughout, X is a Banach space, $M \subseteq X$ is nonempty, closed and convex, $T : M \to M$, and $Fix(T) = \{p \in M : Tp = p\}$.

Definition 2.1. T is *nonexpansive* if $\|Tu - Tv\| \le \|u - v\|$ for all $u, v \in M$. T satisfies *condition (C)* (is a *Suzuki mapping*) [20] if $\frac{1}{2}\|u - Tu\| \le \|u - v\| \Rightarrow \|Tu - Tv\| \le \|u - v\|$ for all $u, v \in M$. T is *generalized* α*-nonexpansive* [17] if there is $\alpha \in [0, 1)$ such that

$$\tfrac{1}{2}\|u - Tu\| \le \|u - v\| \Longrightarrow \|Tu - Tv\| \le \alpha\|Tu - v\| + \alpha\|Tv - u\| + (1 - 2\alpha)\|u - v\|$$

for all $u, v \in M$. Every Suzuki mapping is generalized 0-nonexpansive, i.e. the case $\alpha = 0$ of (1).

Definition 2.2. X is *uniformly convex* if for every $\varepsilon \in (0, 2]$ there is $\delta > 0$ such that $\|u\|, \|v\| \le 1$ and $\|u - v\| \ge \varepsilon$ imply $\left\|\frac{u+v}{2}\right\| \le 1 - \delta$. X has *Opial's property* [16] if $u_k \rightharpoonup w$ implies $\limsup_k \|u_k - w\| < \limsup_k \|u_k - y\|$ for every $y \neq w$. T satisfies *condition (I)* [19] if there is a nondecreasing $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r > 0$, and $\|u - Tu\| \ge f(\mathrm{dist}(u, Fix(T)))$ for all $u \in M$. For a bounded sequence $\{u_k\}$ the *asymptotic radius* relative to M is $\inf_{y \in M} \limsup_k \|u_k - y\|$ and the *asymptotic center* is the set of

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points of M attaining it; in a uniformly convex space this
set is a single point [8].

Lemma 2.3 (Schu [18]). *Let* X *be uniformly convex,* $\{t_k\} \subset [a, b]$ *with* $0 < a \le b < 1$, *and* $\{x_k\}, \{y_k\} \subset X$ *with* $\limsup \|x_k\| \le c$, $\limsup \|y_k\| \le c$ *and* $\lim \|t_k x_k + (1 - t_k) y_k\| = c$ *for some* $c \ge 0$. *Then* $\lim \|x_k - y_k\| = 0$

Lemma 2.4. *Let* $T : M \to M$ *be generalized* α*-nonexpansive. Then:*
if $\mathrm{Fix}(T) \neq \emptyset$, *then* $\|Tu - p\| \le \|u - p\|$ *for all* $u \in M$, $p \in \mathrm{Fix}(T)$;
$\mathrm{Fix}(T)$ *is closed;*
for all $u, v \in M$,

$$\|u - Tv\| \le \mu \|u - Tu\| + \|u - v\|, \quad \mu = \frac{3 + \alpha}{1 - \alpha}$$

(Property (2) *is Garc´ıa-Falset–Llorens-Fuster–Suzuki's condition (E) [7] with constant* μ.*)*

Proof. (a) Take $u := p$ and v arbitrary in (1). The premise holds because $\frac{1}{2}\|p - Tp\| = 0$. Hence
$\|p - Tv\| \le \alpha\|p - v\| + \alpha\|Tv - p\| + (1 - 2\alpha)\|p - v\|$, i.e. $(1 - \alpha)\|Tv - p\| \le (1 - \alpha)\|v - p\|$.
Let $p_j \in \mathrm{Fix}(T)$, $p_j \to p$. Apply (1) with $u = p_j$, $v = p$ (the premise holds trivially):
$\|p_j - Tp\| \le \alpha\|p_j - p\| + \alpha\|Tp - p_j\| + (1 - 2\alpha)\|p_j - p\|$. Letting $j \to \infty$ gives $\|p - Tp\| \le \alpha\|Tp - p\|$, so $Tp = p$.

Step 0. Taking $v = Tu$ in (1) (premise trivially true) gives $\|Tu - T^2u\| \le \alpha\|T^2u - u\| + (1 - 2\alpha)\|u - Tu\| \le \alpha\|T^2u - Tu\| + (1 - \alpha)\|u - Tu\|$, hence $\|Tu - T^2u\| \le \|u - Tu\|$.

Step 1 (dichotomy). For all u, v, either $\frac{1}{2}\|u - Tu\| \le \|u - v\|$ or $\frac{1}{2}\|Tu - T^2u\| \le \|Tu - v\|$. Otherwise, by Step 0, $\|u - Tu\| \le \|u - v\| + \|v - Tu\| < \frac{1}{2}\|u - Tu\| + \frac{1}{2}\|Tu - T^2u\| \le \|u - Tu\|$,
a contradiction.

Case A: $\frac{1}{2}\|u - Tu\| \le \|u - v\|$. Then by (1) and the triangle inequality,
$\|u - Tv\| \le \|u - Tu\| + \|Tu - Tv\| \le (1 + \alpha)\|u - Tu\| + (1 - \alpha)\|u - v\| + \alpha\|u - Tv\|$,
so $\|u - Tv\| \le \frac{1+\alpha}{1-\alpha}\|u - Tu\| + \|u - v\|$, which implies (2).

Case B: $\frac{1}{2}\|Tu - T^2u\| \le \|Tu - v\|$. Apply (1) to the pair (Tu, v): $\|T^2u - Tv\| \le \alpha\|T^2u - v\| + \alpha\|Tv - Tu\| + (1 - 2\alpha)\|Tu - v\|$.
Using $\|T^2u - v\| \le \|u - Tu\| + \|Tu - v\|$ (Step 0), $\|Tv - Tu\| \le \|Tv - u\| + \|u - Tu\|$ and $\|Tu - v\| \le \|u - Tu\| + \|u - v\|$ we get $\|T^2u - Tv\| \le (1 + \alpha)\|u - Tu\| + (1 - \alpha)\|u - v\| + \alpha\|u - Tv\|$. Therefore
$\|u - Tv\| \le \|u - Tu\| + \|Tu - T^2u\| + \|T^2u - Tv\| \le (3 + \alpha)\|u - Tu\| + (1 - \alpha)\|u - v\| + \alpha\|u - Tv\|$,
which is (2).

Lemma 2.5 (Demiclosedness). *Let* X *have Opial's property and let* T *be generalized* α*-nonexpansive. If* $u_k \rightharpoonup z$ *and* $\|u_k - Tu_k\| \to 0$, *then* $Tz = z$.

Proof. By (2), $\limsup \|u_k - Tz\| \le \mu \lim \|u_k - Tu_k\| + \limsup \|u_k - z\| = \limsup \|u_k - z\|$. If $Tz \neq z$, Opial's property would give the strict inequality $\limsup \|u_k - z\| < \limsup \|u_k - Tz\|$, a contradiction.

The JK scheme and its convergence

Definition 3.1 (JK scheme). For $u_1 \in M$ and sequences $\{\xi_k\}, \{\varpi_k\} \subset [0, 1]$ define

$$w_k = (1 - \xi_k)u_k + \xi_k Tu_k, \quad v_k = Tw_k,$$
$$u_{k+1} = T\left((1 - \varpi_k)v_k + \varpi_k Tv_k\right).$$

Remark 3.2. Scheme (3) is the JK scheme of Ahmad et al. [4] in the form displayed in [15, Eq. (2.1)], with our notation for the sequences. We could not access the display of the defining equation in the original article (Eq. (12) there), so the reader should compare (3) with it. One iteration needs four evaluations of T: Tu_k, Tw_k, Tv_k and the outer T. We write $y_k := (1-\varpi_k)v_k + \varpi_k T v_k$, so $u_{k+1} = Ty_k$.

Proposition 3.3 (Monotonicity)**.** *Let T be generalized α-nonexpansive with $\mathrm{Fix}(T) \neq \emptyset$, and let $\{u_k\}$ be generated by (3). Then for every $p \in \mathrm{Fix}(T)$,*

$$\|u_{k+1}-p\| \le \|y_k-p\| \le \|v_k-p\| \le \|w_k-p\| \le \|u_k-p\|.$$

In particular $\lim_k \|u_k-p\|$ exists and $\{u_k\}$ is bounded.

Proof. By Lemma 2.4(a) and convexity of the norm, $\|w_k-p\| \le (1-\xi_k)\|u_k-p\| + \xi_k\|Tu_k-p\| \le \|u_k-p\|$; $\|v_k-p\| = \|Tw_k-p\| \le \|w_k-p\|$; $\|y_k-p\| \le (1-\varpi_k)\|v_k-p\| + \varpi_k\|Tv_k-p\| \le \|v_k-p\|$; and $\|u_{k+1}-p\| = \|Ty_k-p\| \le \|y_k-p\|$. A nonincreasing sequence of nonnegative reals converges.

Theorem 3.4 (Approximate fixed point sequence)**.** *Let X be uniformly convex, T generalized α-nonexpansive, and $\xi_k \in [a,b] \subset (0,1)$. Then $\mathrm{Fix}(T) \neq \emptyset$ if and only if $\{u_k\}$ is bounded and $\lim_k \|u_k - Tu_k\| = 0$.*

Proof. ($\Rightarrow$) Let $p \in \mathrm{Fix}(T)$ and $r = \lim \|u_k - p\|$ (Proposition 3.3). Since $\|u_{k+1}-p\| \le \|w_k-p\| \le \|u_k-p\|$, also $\|w_k-p\| \to r$. Put $x_k = u_k - p$, $y'_k = Tu_k - p$, $t_k = 1-\xi_k \in [1-b, 1-a] \subset (0,1)$.

Then $\limsup\|x_k\| \le r$, $\limsup\|y'_k\| \le r$ (Lemma 2.4(a)) and $\|t_k x_k + (1-t_k)y'_k\| = \|w_k - p\| \to r$.

Lemma 2.3 gives $\|u_k - Tu_k\| = \|x_k - y'_k\| \to 0$.

($\Leftarrow$) Let z be the asymptotic center of $\{u_k\}$ relative to M. By (2), $\limsup\|u_k - Tz\| \le \mu \lim\|u_k - Tu_k\| + \limsup\|u_k - z\| = \limsup\|u_k - z\|$, so Tz is also an asymptotic center. Uniqueness gives $Tz = z$.

Theorem 3.5 (Weak convergence)**.** *Let X be uniformly convex with Opial's property, T generalized α-nonexpansive with $\mathrm{Fix}(T) \neq \emptyset$, $\xi_k \in [a,b] \subset (0,1)$, $\varpi_k \in [0,1]$. Then $\{u_k\}$ converges weakly to a fixed point of T.*

Proof. X is reflexive and $\{u_k\}$ is bounded, so it has a weakly convergent subsequence $u_{k_j} \rightharpoonup z$, and $z \in M$ because M is closed and convex, hence weakly closed. By Theorem 3.4 and Lemma 2.5, $z \in \mathrm{Fix}(T)$. Suppose another subsequence $u_{m_j} \rightharpoonup z'$ with $z' \neq z$; likewise $z' \in \mathrm{Fix}(T)$. Since $\lim\|u_k - z\|$ and $\lim\|u_k - z'\|$ exist (Proposition 3.3), Opial's property gives

$$\lim_j \|u_{k_j} - z\| = \lim \|u_k - z\| < \lim_j \|u_{k_j} - z'\| = \lim \|u_k - z'\| = \lim_j \|u_{m_j} - z'\| < \lim_j \|u_{m_j} - z\| = \lim \|u_k - z\|,$$

a contradiction. All weakly convergent subsequences thus share one limit, and the whole (bounded) sequence converges weakly to it. □

Theorem 3.6 (Strong convergence)**.** *Let X be uniformly convex, T generalized α-nonexpansive with $\mathrm{Fix}(T) \neq \emptyset$, $\xi_k \in [a,b] \subset (0,1)$, $\varpi_k \in [0,1]$. Then $\{u_k\}$ converges strongly to a fixed point of T in each of the following cases:*

M is compact;

$\liminf_k \mathrm{dist}(u_k, \mathrm{Fix}(T)) = 0$;

T satisfies condition (I).

Proof. (b) By Proposition 3.3, $\mathrm{dist}(u_{k+1}, \mathrm{Fix}(T)) \le \mathrm{dist}(u_k, \mathrm{Fix}(T))$, so the limit

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exists and equals 0. Given $\varepsilon > 0$ choose N and $p \in \mathrm{Fix}(T)$ with $\|u_N - p\| < \varepsilon/2$. For $m, n \geq N$,
$\|u_m - u_n\| \leq \|u_m - p\| + \|u_n - p\| \leq 2\|u_N - p\| < \varepsilon$. So $\{u_k\}$ is Cauchy and converges to some
$q \in M$; then $\mathrm{dist}(q, \mathrm{Fix}(T)) = 0$ and $\mathrm{Fix}(T)$ is closed (Lemma 2.4(b)), so $q \in \mathrm{Fix}(T)$.
Some subsequence $u_{k_j} \to z \in M$. By (2), $\|u_{k_j} - Tz\| \leq \mu \|u_{k_j} - Tu_{k_j}\| + \|u_{k_j} - z\| \to 0$ (Theorem 3.4), so $Tz = z$. Since $\lim \|u_k - z\|$ exists and is 0 along a subsequence, $u_k \to z$.
(c) By Theorem 3.4, $f(\mathrm{dist}(u_k, \mathrm{Fix}(T))) \leq \|u_k - Tu_k\| \to 0$. The nonincreasing sequence $\mathrm{dist}(u_k, \mathrm{Fix}(T))$ has a limit ℓ; if $\ell > 0$ then $f(\mathrm{dist}(u_k, \mathrm{Fix}(T))) \geq f(\ell) > 0$ for all k, a contradiction. Hence $\ell = 0$ and (b) applies.
Remark 3.7. Existence of a fixed point is assumed above. Existence results for generalized α-nonexpansive mappings on bounded closed convex sets are available in [17]; in the example of Section 4 the fixed point is found directly. The parameter ϖ_k enters the proofs only through $\varpi_k \in [0, 1]$, because the last two steps of (3) are quasi-nonexpansive whatever ϖ_k is.

Proposition 3.8 (Contractions)**.** *Let* T *be a* δ*-contraction* ($\delta < 1$) *with fixed point* p*. For any*
$\xi_k, \varpi_k \in [0, 1]$,
$\|u_{k+1} - p\| \leq \delta^2\, 1 - (1-\delta)\xi_k \;\; 1 - (1-\delta)\varpi_k \; \|u_k - p\| \leq \delta^2 \|u_k - p\|$.
Hence $\|u_{k+1} - p\| \leq \delta^{2k} \|u_1 - p\|$.
Proof. $\|w_k - p\| \leq [1 - (1-\delta)\xi_k]\|u_k - p\|$; $\|v_k - p\| \leq \delta \|w_k - p\|$; $\|y_k - p\| \leq [1 - (1-\delta)\varpi_k]\|v_k - p\|$;
$\|u_{k+1} - p\| \leq \delta \|y_k - p\|$. Multiply. □

A new example that is not a Suzuki mapping

Example 4.1. Let $X = \mathbb{R}$ with the absolute value and $M = [0, 4]$. Define $T : M \to M$ by

$$T(x) = \begin{cases} \frac{3}{5}x + \frac{2}{5}, & 0 \leq x \leq 2, \\ \frac{6}{5}, & 2 < x \leq 4. \end{cases}$$

Write $L = [0, 2]$ and $U = (2, 4]$. Then $T(M) \subseteq [\frac{2}{5}, \frac{8}{5}] \subset L$. See Figure 1.

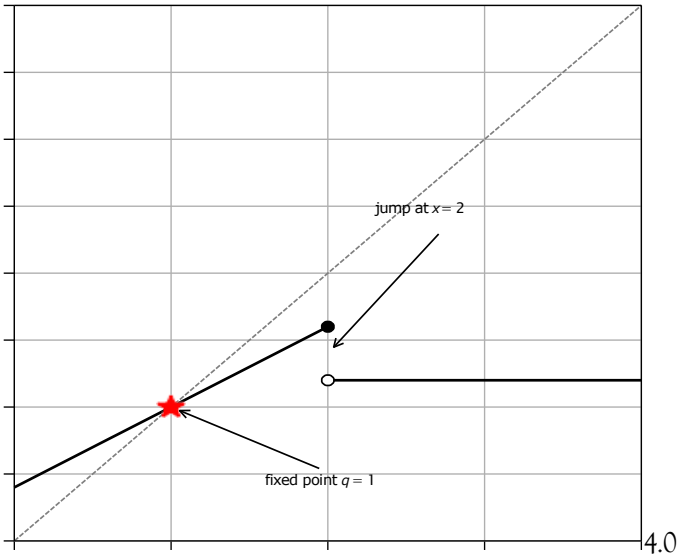


Figure 1: The map T of Example 4.1: affine with slope 3/5 on [0, 2], constant 6/5 on (2, 4], with a jump at x = 2.

Proposition 4.2. *For the map T of Example 4.1:*

Fix(T) = {1};

T *is generalized* α*-nonexpansive with* $\alpha = \frac{1}{5}$*, and* $\alpha = \frac{1}{5}$ *is the smallest admissible value;*

T *is not nonexpansive and does not satisfy condition (C), so it is not a Suzuki mapping. Consequently Theorems 3.5 and 3.6 apply to* T *on* M = [0, 4] *(compact and convex in* R*, which is uniformly convex with Opial's property).*

Proof. (a) On L, T(x) = x iff x = 1. On U, T(x) = 6/5 $\notin$ U, so there is no fixed point in U.

(c) Take $x = 2$, $y = \frac{11}{5}$. Then $Tx = \frac{8}{5}$, $Ty = \frac{6}{5}$, $\frac{1}{2}|x - Tx| = \frac{1}{5} = |x - y|$, but $|Tx - Ty| = \frac{2}{5} > \frac{1}{5} = |x - y|$. So condition (C) fails at this pair, and nonexpansiveness fails as well.

Set $\alpha = \frac{1}{5}$ and $\Phi(x, y) := \frac{1}{5}|Tx - y| + \frac{1}{5}|Ty - x| + \frac{3}{5}|x - y| - |Tx - Ty|$. We must show $\Phi(x, y) \geq 0$ whenever $\frac{1}{2}|x - Tx| \leq |x - y|$.

Case 1: $x, y \in L$. Since $Tx - Ty = \frac{3}{5}(x - y)$, we have $|Tx - y| + |Ty - x| \geq |(Tx - y) + (x - Ty)| = |(x - y) + (Tx - Ty)| = \frac{8}{5}|x - y|$. Hence $\Phi \geq \frac{8}{5}|x - y| + \frac{3}{5}|x - y| - \frac{3}{5}|x - y| \geq 0$ (no premise needed).

Case 2: $x, y \in U$. Then $Tx = Ty$ and every term of Φ other than $-|Tx - Ty| = 0$ is nonnegative.

Case 3: $x \in L$, $y \in U$, *premise holds.* Put $s = x - 1 \in [-1, 1]$ and $t = y - 1 \in (1, 3]$. Then $Tx - 1 = \frac{3}{5}s$, $Ty - 1 = \frac{1}{5}$, $|x - Tx| = \frac{2}{5}|s|$, so the premise reads $t \geq s + \frac{1}{2}|s|$. Because $Tx \leq \frac{8}{5} < y$, $|Tx - y| = t - \frac{3}{5}s$; also $|x - y| = t - s$, $|Ty - x| = |\frac{1}{5} - s|$ and $|Tx - Ty| = \frac{1}{5}|3s - 1|$. A direct computation gives

$$25\,\Phi(x, y) = G(t, s) := 20t - 18s + |1 - 5s| - 5|3s - 1|.$$

If $s \leq \frac{1}{5}$: $G = 20t - 8s - 4 \geq 20 - \frac{8}{5} - 4 > 0$ (using $t > 1$, $s \leq \frac{1}{5}$).

If $\frac{1}{5} < s \leq \frac{1}{3}$: $G = 20t + 2s - 6 > 14 > 0$.

If $\frac{1}{3} < s \leq 1$: $G = 20t - 28s + 4$, and the premise gives $t \geq \frac{1}{5}s$, so $G \geq 24s - 28s + 4 = 4(1 - s) \geq 0$.

Case 4: $x \in U$, $y \in L$, *premise holds.* Put $e = x - 1 \in (1, 3]$ and $r = y - 1 \in [-1, 1]$. Then $Tx - 1 = \frac{1}{5}$, $Ty - 1 = \frac{3}{5}r$, $|x - Tx| = e - \frac{1}{5}$, $|x - y| = e - r$, so the premise reads $e \geq 2r - \frac{1}{5}$.

Moreover $|Tx - y| = |\frac{1}{5} - r|$, $|Ty - x| = e - \frac{3}{5}r$ (as $e > 1 \geq \frac{3}{5}r$) and $|Tx - Ty| = \frac{1}{5}|1 - 3r|$. The same computation gives $25\,\Phi(x, y) = G(e, r)$ with G as in (4):

if $r \leq \frac{1}{5}$: $G = 20e - 8r - 4 > 0$; if $\frac{1}{5} < r \leq \frac{1}{3}$: $G = 20e + 2r - 6 > 0$;

if $\frac{1}{3} < r \leq 1$: $G = 20e - 28r + 4 \geq (40r - 4) - 28r + 4 = 12r > 0$, using $e \geq 2r - \frac{1}{5}$.

Thus (1) holds with $\alpha = \frac{1}{5}$. For minimality, at $(x, y) = (2, \frac{11}{5})$, where the premise holds, the inequality in (1) with general α reads $\frac{2}{5} \leq \frac{3}{5}\alpha + \frac{4}{5}\alpha + \frac{1}{5}(1 - 2\alpha) = \alpha + \frac{1}{5}$, which forces $\alpha \geq \frac{1}{5}$.

Remark 4.3. On L the map is affine with slope 3/5 and fixed point 1. Every scheme in Section 5 produces convex combinations of previous iterates and their images, and L is convex with $T(L) \subseteq L$. Hence *a run started in* L *never leaves* L *and is exactly the run of that scheme on the linear*

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contraction $x \mapsto 1 + \frac{3}{5}(x - 1)$. The jump only influences runs started in U, and only during the first iterations, since $T(M) \subseteq L$. We use this structure in Section 5.2, and we point out that it limits what this single example can show.

As a sanity check separate from the proof, we verified (1) with $\alpha = \frac{1}{5}$ in exact rational arithmetic on all ordered pairs of a grid of step 1/60 on [0, 4] enriched with points within 10^{-4} of the jump (128,881 pairs, 91,877 satisfying the premise; no violation), and with floating point on a finer grid, where the smallest admissible α came out as 0.2, attained at $(2, \frac{11}{5})$. Such grid checks
are consistency checks only.

Numerical experiments

Design

Schemes. With $u_1 \in M$ and sequences $\{\alpha_k\}$, $\{\beta_k\}$, $\{\gamma_k\} \subset (0, 1)$, the compared schemes are, in the forms of the original papers [12, 10, 13, 2, 1, 21]:

Mann: $u_{k+1} = (1 - \alpha_k)u_k + \alpha_k T u_k$;

Ishikawa: $v_k = (1 - \beta_k)u_k + \beta_k T u_k$, $u_{k+1} = (1 - \alpha_k)u_k + \alpha_k T v_k$; Noor: $w_k = (1 - \gamma_k)u_k + \gamma_k T u_k$, $v_k = (1 - \beta_k)u_k + \beta_k T w_k$, $u_{k+1} = (1 - \alpha_k)u_k + \alpha_k T v_k$;

Agarwal: $v_k = (1 - \beta_k)u_k + \beta_k T u_k$, $u_{k+1} = (1 - \alpha_k)T u_k + \alpha_k T v_k$; Abbas–Nazir: $w_k = (1 - \gamma_k)u_k + \gamma_k T u_k$, $v_k = (1 - \beta_k)T u_k + \beta_k T w_k$, $u_{k+1} = (1 - \alpha_k)T v_k + \alpha_k T w_k$;

Thakur: $w_k = (1 - \gamma_k)u_k + \gamma_k T u_k$, $v_k = T((1 - \beta_k)u_k + \beta_k w_k)$, $u_{k+1} = T v_k$;

JK: scheme (3) with $\xi_k = \alpha_k$, $\varpi_k = \beta_k$.

The numbers of distinct evaluations of T per iteration are 1, 2, 3, 2, 3, 3, 4 respectively.

Protocol. All runs use the map of Example 4.1 with known fixed point $q = 1$, and stop at the first k with $|u_{k+1} - 1| < \mathrm{tol}$; we report k as the number of iterations (updates performed) together with the number of evaluations of T. We use $\mathrm{tol} \in \{10^{-10}, 10^{-15}\}$ and 50-digit arithmetic (Python, mpmath), because a tolerance of 10^{-15} is near the resolution of double precision. Starting points are $u_1 \in \{0.1, 1.5, 2.0, 2.5, 3.9\}$: two in L away from the jump, the jump point 2 itself (which belongs to L), and two in U. Six parameter settings are used, with $\alpha_k = \beta_k = \gamma_k =: s_k$: constant 0.2, 0.5, 0.8; increasing $s_k = 0.5 + 0.4(1 - \frac{1}{k+1}) \to 0.9$; decreasing $s_k = 0.5 - 0.4(1 - \frac{1}{k+1}) \to 0.1$; and oscillating $s_k = 0.5 \pm 0.2 \sin k$. All lie in a compact subinterval of (0, 1), as Theorems 3.5–3.6 require. Following [3], a scheme converges faster than another if it reaches the same accuracy in fewer iterations with the same parameters and starting point. This gives $2 \times 6 \times 5 = 60$ experiments per scheme. The sequences used in [3, 4] are not included because we could not access them; this is a limitation.

Exact one-step factors on the affine part

Proposition 5.1. *Let* $\delta = \frac{3}{5}$, *let the parameters be constant,* $(\alpha_k, \beta_k, \gamma_k) \equiv (a, b, g)$, *and* $u_1 \in L$. *Then* $e_k := u_k - 1$ *satisfies* $e_{k+1} = \rho_S e_k$ *for every scheme* S, *where, with* $x = 1 - \delta$,

$\rho_{\mathrm{Mann}} = 1 - xa$,

$\rho_{\mathrm{AN}} = (1 - a)\delta^2(1 - xbg) + a\delta(1 - xg)$,

Proof. By Remark 4.3, all iterates stay in L, where $T(u) - 1 = \delta(u - 1)$. Every scheme is then a composition of linear maps of the deviations from 1, and one propagates the deviation through the steps. For instance, for JK: $w - 1 = (1 - xa)e$, $v - 1 = \delta(w - 1)$, $y - 1 = (1 - xb)(v - 1)$, $u_+ - 1 = \delta(y - 1)$. The remaining schemes are treated in the same way; the routine computations are omitted, and the resulting factors are confirmed by the counts in Table 1.

Proposition 5.2. *If* $a = b = g = c \in (0, 1)$ *and* $\delta \in (0, 1)$, *then* $\rho_{\mathrm{JK}} < \rho_S$ *for each of the other six schemes* S.

Proof. Put $P = 1 - xc \in (0, 1)$ and $Q = 1 - xc^2$. Then $\rho_{\mathrm{JK}} = \delta^2 P^2$, and $Q - P^2 = xc(2 - c - xc) > 0$. Mann: $\rho = P > P^2 > \rho_{\mathrm{JK}}$. Ishikawa: $\rho = (1 - c) \cdot 1 + c \cdot \delta P \geq \delta P > \delta^2 P^2$, since $\delta P < 1$. Noor: $\rho = (1 - c) + c\delta V$ with $V = (1 - c) + c\delta P \geq \delta P$, so $\rho \geq (1 - c) + c\delta^2 P \geq \delta^2 P > \delta^2 P^2$. Agarwal: $\rho = \delta Q > \delta P^2 > \delta^2 P^2$. Abbas–Nazir:

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$\rho = (1-c)\delta^2 Q + c\delta P > (1-c)\delta^2 P^2 + c\delta^2 P^2$, using $Q > P^2$ and $\delta P > \delta^2 P^2$. Thakur: $\rho = \delta^2 Q > \delta^2 P^2$.

Table 1 lists the factors for $c \in \{0.2, 0.5, 0.8\}$ and compares the predicted number of iterations $\lceil \ln(10^{-15}/e_1)/\ln \rho \rceil$ with the observed number, for $u_1 = 0.1$ ($e_1 = 0.9$). They agree in all 21 cases, so for starts in L the numerical experiment is fully explained by Proposition 5.1.

Table 1: Exact contraction factor ρ per iteration, and predicted/observed iterations to reach 10^{-15} from
$u_1 = 0.1$ (all parameters equal to c).

c		Mann	Ishikawa	Noor	Agarwal	Abbas–Nazir	Thakur	JK
0.2	ρ	0.9200	0.9104	0.9092	0.5904	0.3938	0.3542	0.3047
	pred./obs.	413/413	367/367	362/362	66/66	37/37	34/34	29/29
0.5	ρ	0.8000	0.7400	0.7220	0.5400	0.4020	0.3240	0.2304
	pred./obs.	155/155	115/115	106/106	56/56	38/38	31/31	24/24
0.8	ρ	0.6800	0.5264	0.4527	0.4464	0.3800	0.2678	0.1665
	pred./obs.	90/90	54/54	44/44	43/43	36/36	27/27	20/20

Results

Table 2 gives iterations (and evaluations of T) for the constant setting 0.5 and tolerance 10^{-15} for all five starting points; Figure 2 shows the error histories. For JK and Thakur the counts vary by at most one iteration across the five starting points (in all 12 tolerance–setting combinations), and the runs started in U enter L after the first iteration, since the last operation of both schemes is an application of T and $T(M) \subseteq L$. The slower schemes depend more on the starting point, because the initial error differs (Mann: 152 to 157 iterations in Table 2, and up to 28 iterations of variation in the slowest settings). Table 3 shows the iteration counts for all six parameter settings and both tolerances (start 2.5), and Table 4 the corresponding evaluation counts.

Table 2: Iterations (evaluations of T) to reach $|u_k - 1| < 10^{-15}$; constant parameters 0.5. Bold: smallest in the row.

u_1	Mann	Ishikawa	Noor	Agarwal	Abbas–Nazir	Thakur	JK
0.1	155 (155)	115 (230)	106 (318)	56 (112)	38 (114)	31 **(93)**	**24** (96)
1.5	152 (152)	113 (226)	104 (312)	55 (110)	38 (114)	31 **(93)**	**24** (96)
2.0	155 (155)	115 (230)	107 (321)	57 (114)	38 (114)	31 **(93)**	**24** (96)
2.5	156 (156)	116 (232)	107 (321)	56 (112)	38 (114)	30 **(90)**	**24** (96)
3.9	157 (157)	117 (234)	108 (324)	55 (110)	37 (111)	30 **(90)**	**23** (92)

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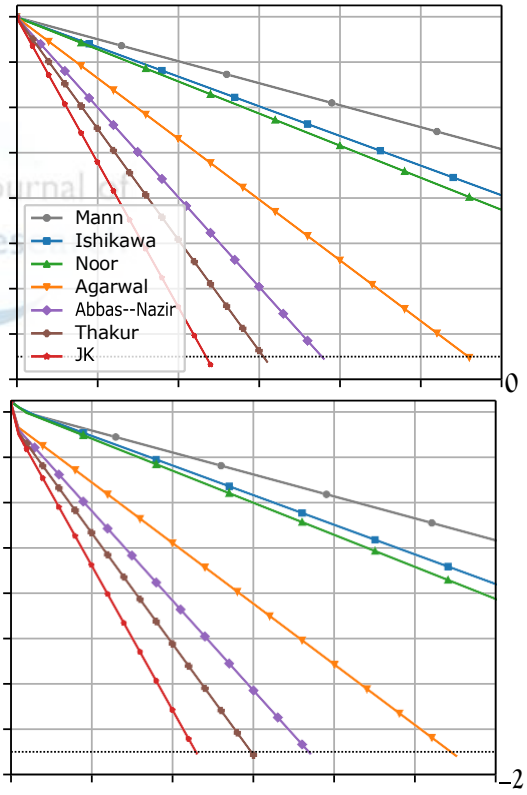


Figure 2: Error histories $\log_{10}|u_k - 1|$, constant parameters 0.5. Dotted line: tolerance 10^{-15}.

Sensitivity to the parameters. Figure 3 shows, for constant parameters c ∈ {0.1, . . . , 0.9} and start 3.9, the iterations and evaluations needed by each scheme. Mann, Ishikawa and Noor are very sensitive to small c: the counts at c = 0.1 lie between 806 and 858, against between 33 and 77 at c = 0.9. JK, Thakur and Abbas–Nazir are far less sensitive. For JK the count falls from 31 at c = 0.1 to 18 at c = 0.9. Figure 4 shows that for JK with independent constants (ξ, ϖ) the iteration count is non-increasing in both, from 32 to 18 (start 0.1).

Computation time. On a single machine (float64, tolerance 10^{-10}, constant parameters 0.5, start 2.5, median of 25 repetitions of 200 runs), the times were as in Table 5. They follow the number of iterations and evaluations and are machine dependent, so we regard them as a rough indication only.

Discussion

Research question 1: does JK need fewer iterations? On this example, yes, in every one of the 60 experiments (six settings, five starts, two tolerances): JK had the strictly smallest iteration count. The ratio of Thakur's count to JK's ranged from about 1.10 to 1.38. For starts in L and constant equal parameters this is an exact statement about contraction factors (Propositions 5.1 and 5.2), not just a numerical observation; for the non-constant settings it is a numerical observation.

Cost per iteration changes the picture. JK evaluates T four times per iteration and Thakur's scheme three times. Counting evaluations, Thakur's scheme needed strictly fewer than JK in 47 of the 60 experiments, JK strictly fewer in 11, and the two tied in 2. The explanation is visible

Table 3: Iterations for each parameter setting and tolerance, start $u_1 = 2.5$. Bold: smallest in the row.

tol	setting	Mann	Ishikawa	Noor	Agarwal	Abbas–Nazir	Thakur	JK
10^{-10}	const. 0.2	278	248	245	42	24	22	**19**
10^{-10}	const. 0.5	104	78	72	37	26	20	**16**
10^{-10}	const. 0.8	59	36	30	28	24	18	**13**
10^{-10}	incr. $0.5 \to 0.9$	54	32	25	27	24	18	**13**
10^{-10}	decr. $0.5 \to 0.1$	543	512	510	43	24	22	**20**
10^{-10}	oscill.	102	73	66	36	25	21	**16**
10^{-15}	const. 0.2	416	371	366	64	36	33	**29**
10^{-15}	const. 0.5	156	116	107	56	38	30	**24**
10^{-15}	const. 0.8	89	54	44	43	36	27	**20**
10^{-15}	incr. $0.5 \to 0.9$	80	46	36	40	35	26	**19**
10^{-15}	decr. $0.5 \to 0.1$	824	775	773	65	36	33	**30**
10^{-15}	oscill.	152	110	100	55	38	31	**24**

Table 4: Evaluations of T to reach 10^{-15} for the settings of Table 3, start $u_1 = 2.5$. Bold: smallest in the row.

setting	Mann	Ishikawa	Noor	Agarwal	Abbas–Nazir	Thakur	JK
const. 0.2	416	742	1098	128	108	**99**	116
const. 0.5	156	232	321	112	114	**90**	96
const. 0.8	89	108	132	86	108	81	**80**
incr. $0.5 \to 0.9$	80	92	108	80	105	78	**76**
decr. $0.5 \to 0.1$	824	1550	2319	130	108	**99**	120
oscill.	152	220	300	110	114	**93**	96

in the exact factors: for constant parameters c and $\delta = \frac{3}{5}$, the per-evaluation rates are $\rho^{1/4}$ and $\rho^{1/3}$, and these cross at $c^* \approx 0.597$. For $c < c^*$ Thakur's scheme is cheaper, for $c > c^*$ JK is (Figure 3, right). For example at $c = 0.5$: $\rho^{1/4} \approx 0.693$ against $\rho^{1/3} \approx 0.687$. Whether "faster" is measured per iteration or per evaluation should therefore be stated explicitly when schemes with different costs are compared.

Research question 2: sensitivity. For JK the number of iterations depends moderately on the parameters (from 31 to 18 over $c \in [0.1, 0.9]$ for start 3.9), is non-increasing in both ξ and ϖ for this example, and is essentially independent of the starting point (counts differ by at most one). The larger the constant parameter, the fewer the iterations, for every scheme except Abbas–Nazir, whose count is almost flat (between 35 and 38). This is consistent with the factors of Proposition 5.1: on the affine part the parameters only enter through terms such as $1 - xa$.

Limitations. There is a single example, and by Remark 4.3 its behaviour is linear on the part L where runs spend almost all their time. It therefore does not test the schemes in a genuinely nonlinear or infinite-dimensional setting, nor over a family of maps with different Lipschitz behaviour.

The parameter sequences used in the

original JK papers could not be reproduced here (Section 5.1).

The 10^{-15} stopping rule uses the known fixed point, which is unavailable in practice; practical codes would use $|u_{k+1} - u_k|$ or the residual $|u_k - T u_k|$.

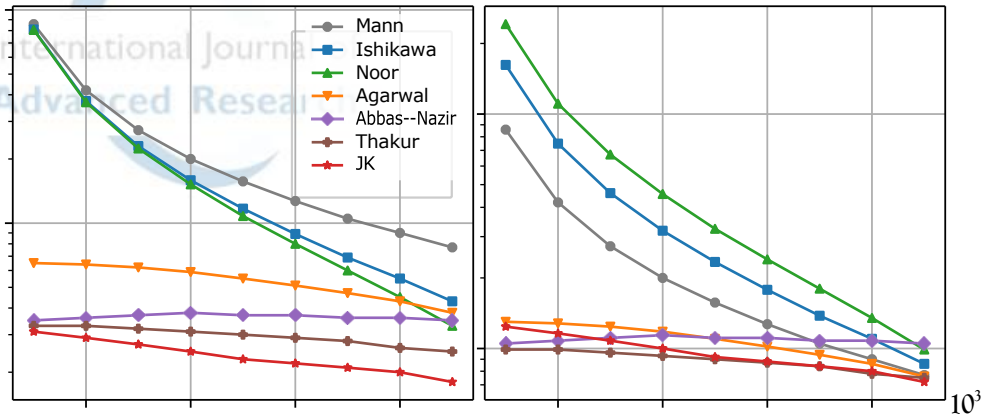


Figure 3: Iterations (left) and evaluations of T (right) to reach 10^{-15} versus the constant parameter c; start $u_1 = 3.9$.

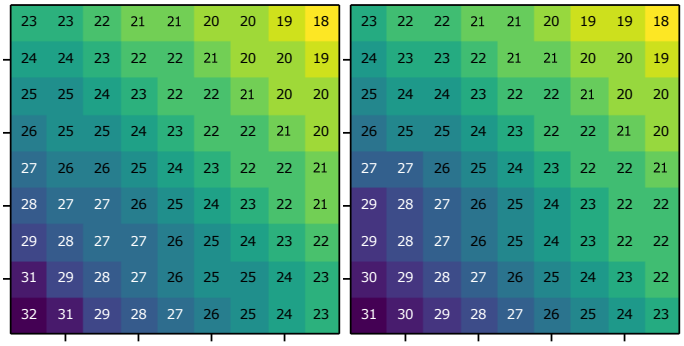


Figure 4: Iterations of JK to reach 10^{-15} for constant parameters (ξ, ϖ). Left: start 0.1; right: start 2.5.

We did not reproduce the numbers in [3, 4]; the faster-than-Thakur claim there is consistent with our per-iteration findings, while those papers compare iteration counts, so our per-evaluation conclusion is not in conflict with them.

Conclusion

We gave complete proofs of the convergence of the JK scheme for generalized α-nonexpansive mappings in uniformly convex spaces, introduced a new non-Suzuki example with a proof of its optimal constant $\alpha = \frac{1}{5}$, and compared JK with six classical schemes over 60 experiments. JK converges in strictly fewer iterations than every other scheme in all of them, and, for constant equal parameters on the affine part of the example, this is explained exactly by a contraction factor that is smallest for JK. Once the four evaluations per iteration are taken into account, Thakur's three-step scheme is cheaper in most experiments; JK is the cheaper scheme only for parameters above about 0.6. Natural continuations are a family of test maps that are genuinely nonlinear, inclusion of the M iteration [23] and of later schemes such as those in [14], and the stability and data-dependence

questions of [22] in the present cost-aware framework.

Table 5: Iterations and run time (microseconds per run) for tolerance 10^{-10}.

Mann		Ishikawa	Noor	Agarwal	Abbas–Nazir	Thakur	JK
Iterations	104	78	72	37	26	20	16
Time (µs)	32	35	45	21	16	11	10

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Reproducibility

All numbers in the tables and figures come from the accompanying Python scripts (schemes.py, run experiments.py, make artifacts.py, verify example.py), which use mpmath with 50 dig-its.

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