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ADVANCED GENERALIZATIONS OF CONVEX FUNCTION INEQUALITIES: IMPLICATIONS FOR HIGH-ORDER DIVERGENCE AND ENTROPY ESTIMATION IN INFORMATION THEORY USING INTERPOLATING POLYNOMIALS

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Abstract
The inequalities involving convex function have many applications in optimization and in recent years it has helped in estimating many entropies and divergences that are used in information theory. In this paper, an inequality constructed by the two inequalities Jensen inequality and Lah-Ribaric inequality is considered. The non-negative difference of the this inequality are used to construct the non-negative difference. Two identities Abel-Gontscharoff and Montgomery identity at a time are used in non-negative differences to construct new identities. Also these identities are used to generalized the inequality for higher order convex function. Furthermore for the sake of application in information theory these generalized results are used to estimate Csiszer divergence, Shannon entropy, Kullback Leibler divergence and Zipf-Mandelbrot laws.

INTRODUCTION

In literature, there are many divergences that are used to find the distance between probability distribution by making an average value, one of them is Csiszár divergence. The Shannon entropy is actually a quantity in information theory that we associate with random variable and it gives an average level of uncertainty, information, or possible outcome of a variable. The researchers working on mathematical inequalities related to convex function have found it very interesting to find the limitations and bounds for the different functionals. Their work motivates us to find the bound for divergence and entropies from information theory. T. Popoviciu [1] was the first to introduce the higher order convexity. It has been observed that some properties are not valid for higher order convex that are valid for convex function and this motivates researchers to study higher-order convexity. A higher-order convex function can be identified by the result that "if $h^{(\lambda)}$ exists and h is λ-convex if and only if $h^{(\lambda)}$ is non-negative".

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There has been significant growth in the study of classical inequalities like the Hölder, Jensen, and others. These inequalities were initially presented in discrete and integral forms, after which numerous generalizations and improvements have been explained (for example, see [2, 3]). Recent research has shown that they are extremely helpful in information theory (for example, see[4]). Researchers have generalized different inequalities for higher-order convex functions using different interpolations like Popoviciu inequality have been generalized by using Lidstone polynomial [5], Hermite interpolating polynomial [6], Fink's identity [7], Taylor's polynomial [8], Abel-Gontscharoff interpolation [5], and Montgomery identity [9]. In [10] Jensen's inequality is given as follows. Let J be an interval in $\mathbb{R}$ and $h: J \to \mathbb{R}$ a convex function. If $z = (z_1, \dots, z_n)$ is any n-tuple in J^n and $\mathbf{q} = (q_1, \dots, q_n)$ a non-negative n-tuple such that $Q_n = \sum_{i=1}^{n} q_i > 0$, then the Jensen's inequality

$$h\left(\frac{1}{Q_n}\sum_{i=1}^{n} q_i\, z_i\right) \le \frac{1}{Q_n}\sum_{i=1}^{n} q_i\, h(z_i), \tag{1}$$

holds. Inequality (1) has remained of great interest among researchers for many years due to its application and its impact on other inequalities because many of the inequalities in literature are the consequence of (1). A lot of refinement of (1) has been constructed. Many researchers have generalized (1) and its refinement (see[11, 12]). S. I. Butt et al. [14] found the application in information theory by using the Jensen-Grüss type inequalities.

The Lah-Ribaric inequality is closely related to Jensen's inequality. In [15, 17] Lah-Ribarič inequality is given as follows,

$$\frac{1}{Q_n}\sum_{i=1}^{n} q_i\, h(z_i) \le \frac{M-\bar{z}}{M-m}h(m) + \frac{\bar{z}-m}{M-m}h(M), \tag{2}$$

which holds when $h: J \to \mathbb{R}$ is a convex function on J, $[m, M]^n \subset J$, $-\infty < m < M < +\infty$, $\mathbf{q}$ is as in (1), $z = (z_1, \dots, z_n)$ is any n-tuple in $[m, M]^n$ and $\bar{z} = \frac{1}{Q_n}\sum_{i=1}^{n} q_i\, z_i$. If h is strictly convex then (1) is strict unless $z_i \in m, M$ for all $i \in \{j: q_j > 0\}$.

The inequality (1) has been researched, and interested readers are able to find numerous related results in more recent research as well as in monographs like [11, 12]. It would be interesting to learn more about how the previously mentioned imbalance is possibly further developed. We also deal with the idea of f-divergences, which evaluates the separation between two probability distributions. One of the most significant is the Csiszár f-divergence, of which specific instances include the Shannon entropy and the Kullback-Leibler divergence. We determine the relationships for the above-mentioned f-divergences. We also examined more closely the Zipf-Mandelbrot law.

2. PRELIMINARIES

Consider the following lemma (see [16]).

Lemma A. Let h be a convex function on an interval J. If $l, m, n, o \in J$ such that $l \le m < n \le o$, then the inequality,

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$$\frac{n-z}{n-m}h(m)+\frac{z-m}{n-m}h(n)\le\frac{o-z}{o-l}h(l)+\frac{z-l}{o-l}h(o), \tag{3}$$

holds for any $z\in[m,n]$.

The result that gives the refinement of the Lah-Ribarič inequality (2) from [17].

Theorem A. Suppose a convex function $h:J\to\mathbb{R}$, where $J\subset\mathbb{R}$ be an interval, $[m, M]\subset J, \mathbf{q}=(q_1,\dots,q_n)$ be a non-negative n-tuple, $\mathbf{z}=(z_1,\dots,z_n)$ is any n-tuple in $[m, M]^n$ and $\bar{z}=\frac{1}{Q_n}\sum_{i=1}^{n}q_i z_i$. Suppose $N_i\subseteq(1,\dots,n)$, $i=1,\dots,m$ where $N_i\cap N_j=\phi$ for $i\neq j$, $\cup_{i=1}^{m}N_i=\{1,\dots,n\}$, $\sum_{j\in N_i}q_j>0$, for $i=1,\dots,m$ and $m_i=\min\{z_j:j\in N_i\}$, $M_i=\max\{x_j:j\in N_i\}$, for $i=1,\dots,m$. Then, the following intervals hold,

$$\begin{aligned}\frac{1}{Q_n}\sum_{i=1}^{n}q_i\,h(z_i) &\le\frac{1}{Q_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i}q_j\right)\left[\frac{M_i-\bar{z}_i}{M_i-m_i}h(m_i)+\frac{\bar{z}_i-m_i}{M_i-m_i}h(M_i)\right]\\ &\le\frac{M-\bar{z}}{M-m}h(m)+\frac{\bar{z}-m}{M-m}h(M),\end{aligned} \tag{4}$$

where if h is a concave function on J, then the inequalities in (4) are reversed.

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Consider the following functionals from (4).

$$\theta_1(h)=\frac{1}{Q_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i}q_j\right)\left[\frac{M_i-\bar{z}_i}{M_i-m_i}h(m_i)+\frac{\bar{z}_i-m_i}{M_i-m_i}h(M_i)\right]-\frac{1}{Q_n}\sum_{i=1}^{n}q_i\,h(z_i), \tag{5}$$

$$\theta_2(h)$$

$$=\frac{M-\bar{z}}{M-m}h(m)+\frac{\bar{z}-m}{M-m}h(M)-\frac{1}{Q_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i}p_j\right)\left[\frac{M_i-\bar{z}_i}{M_i-m_i}h(m_i)+\frac{\bar{z}_i-m_i}{M_i-m_i}h(M_i)\right], \tag{6}$$

$$\theta_3(h)=\frac{M-\bar{z}}{M-m}h(m)+\frac{\bar{z}-m}{M-m}h(M)-\frac{1}{Q_n}\sum_{i=1}^{n}q_i\,h(z_i), \tag{7}$$

If the assumption of Theorem A is valid then

$$\theta_i(h)\ge 0,\ i=1,2,3, \tag{8}$$

and inequalities symbols in (8) are reversed if h is a concave function.

Let $h_1,h_2,h_3:I\to\mathbb{R}$ be three functions with the same assumption of Theorem A and μ_1,μ_2 and μ_3 be any scalars, then

$$\theta_i(\mu_1h_1+\mu_2h_2+\mu_3h_3)=\mu_1\theta_i(h_1)+\mu_2\theta_i(h_2)+\mu_3\theta_i(h_3),\ i=1,2,3. \tag{9}$$

For $n=2$, the Abel-Gontscharoff [18] polynomial is given by

$$h(z)=h(\beta_1)+(z-\beta_1)h'(\beta_2)+\int_{\beta_1}^{\beta_2}G_1(z,v)h''(v)\,dv, \tag{10}$$

where

$$G_1(z,v)=\begin{cases}\beta_1-v, & \beta_1\le v\le z;\\ \beta_1-z, & z\le v\le\beta_2.\end{cases} \tag{11}$$

In [7], the new Green functions are defined as follows.

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$$G_2(z,v)=\begin{cases}z-\beta_2, & \beta_1\le v\le z;\\ v-\beta_2, & z\le v\le \beta_2.\end{cases} \tag{12}$$

$$G_3(z,v)=\begin{cases}z-\beta_1, & \beta_1\le v\le z;\\ v-\beta_1, & z\le v\le \beta_2.\end{cases} \tag{13}$$

$$G_4(z,v)=\begin{cases}\beta_2-v, & \beta_1\le v\le z;\\ \beta_2-z, & z\le v\le \beta_2.\end{cases} \tag{14}$$

Remark A. *Observe that, since*

$$\frac{d^2}{dv^2}G_k(.,v)\ge 0,\ k=1,\dots,4, \tag{15}$$

therefore by using this function in Theorem A we observe

$$\theta_i\big(G_k(.,v)\big)\ge 0,\ k=1,\dots,4,\ i=1,2,3. \tag{16}$$

The authors in [7] also use these new Green functions to construct the following three new identities for twice differentiable functions on $[\beta_1,\beta_2]$.

$$h(z)=h(\beta_2)+(\beta_2-z)h'(\beta_1)+\int_{\beta_1}^{\beta_2}G_2(z,v)h''(v)\,dv, \tag{17}$$

$$h(z)=h(\beta_2)-(\beta_2-\beta_1)h'(\beta_2)+(z-\beta_1)h'(\beta_1)+\int_{\beta_1}^{\beta_2}G_3(z,v)h''(v)\,dv, \tag{18}$$

$$h(z)=h(\beta_1)-(\beta_2-\beta_1)h'(\beta_1)+(\beta_2-z)h'(\beta_2)+\int_{\beta_1}^{\beta_2}G_4(z,v)h''(v)\,dv. \tag{19}$$

In [18], Montgomery's identity is presented as: Let $m\in\mathbb{N}$, $\varphi: I\to\mathbb{R}$ be such that $\varphi^{(m-1)}$ is absolutely continuous, $I\subset\mathbb{R}$ be an open interval, $\beta_1,\beta_2\in I$, $\beta_1<\beta_2$. Then the following identity holds:

$$\varphi(\mu)=\frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2}\varphi(l)\,dl+\sum_{k=0}^{\lambda-2}\frac{\varphi^{(k+1)}(\beta_1)(\mu-\beta_1)^{k+2}}{k!\,(k+2)(\beta_2-\beta_1)}-\sum_{k=0}^{\lambda-2}\frac{\varphi^{(k+1)}(\beta_2)(\mu-\beta_2)^{k+2}}{k!\,(k+2)(\beta_2-\beta_1)}+\frac{1}{(\lambda-1)!}\int_{\beta_1}^{\beta_2}R_\lambda(\mu,l)\varphi^{(\lambda)}(l)\,dl, \tag{20}$$

where

$$R_\lambda(\mu,l)=\begin{cases}-\dfrac{(\mu-l)^\lambda}{\lambda(\beta_2-\beta_1)}+\dfrac{\mu-\beta_1}{\beta_2-\beta_1}(\mu-l)^{\lambda-1}, & \beta_1\le l\le\mu;\\ -\dfrac{(\mu-l)^\lambda}{\lambda(\beta_2-\beta_1)}+\dfrac{\mu-\beta_2}{\beta_2-\beta_1}(\mu-l)^{\lambda-1}, & \mu\le l\le\beta_2.\end{cases} \tag{21}$$

3. GENERALIZATION FOR HIGHER-ORDER CONVEX FUNCTION

In this section, the inequality (4) is generalized for higher order convex function using two interpolations at a time.

Consider the following hypothesis:

($\mathbf{H}_1$) Suppose a function $\varphi\in C^\lambda[\beta_1,\beta_2]$.

($\mathbf{H}_2$) Suppose $z_1,\dots,z_n\in[\beta_1,\beta_2]$ and $q_1,\dots,q_n$ are positive real numbers such that $q_1+\cdots+q_n=1$.

In the following result, new identities are constructed using non-negative differences give in (5)-(7).

Theorem 3.1. *Assume* ($\mathbf{H}_1$) *and* ($\mathbf{H}_2$), *suppose that* $\theta_i(i=1,2,3), G_k(k=1,2,3,4)$ *are the same as*

defined in $(5)-(7)$ *and* $(11)-(14)$ *respectively, then*

$$\theta_i(\varphi) = \sum_{k=0}^{\lambda-4} \frac{1}{k!\,(k+2)(\beta_2-\beta_1)} \int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right)\left[\varphi^{(k+3)}(\beta_1)(v-\beta_1)^{k+2} - \varphi^{(k+3)}(\beta_2)(\right. + \frac{\varphi'(\beta_2)-\varphi'(\beta_1)}{\beta_2-\beta_1} \int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right) dv + \frac{1}{(\lambda-3)!} \int_{\beta_1}^{\beta_2} \varphi^{(\lambda)}(l) \left[\int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right)\right. \tag{22}$$

Proof. Using (10), (17), (18), (19) in (5), (6), and (7), and using property (9) of $\theta_i(\varphi)$, we get

$$\theta_i(\varphi) = \int_{\beta_1}^{\beta_2} \varphi''(v)\theta_i(G_k(.,v)\,dv,\ i = 1,2,3. \tag{23}$$

In (20), make the replacement φ with φ'', and λ with $\lambda-2$, and μ with v, we get

$$\varphi''(v) = \frac{\varphi'(\beta_2)-\varphi'(\beta_1)}{\beta_2-\beta_1} + \sum_{k=0}^{\lambda-4} \frac{\varphi^{(k+3)}(\beta_1)(v-\beta_1)^{k+2}}{k!\,(k+2)(\beta_2-\beta_1)} - \sum_{k=0}^{\lambda-4} \frac{\varphi^{(k+3)}(\beta_2)(v-\beta_2)^{k+2}}{k!\,(k+2)(\beta_2-\beta_1)} + \frac{1}{(\lambda-3)!} \int_{\beta_1}^{\beta_2} R_{\lambda-2}(v,l)\varphi^{(\lambda)}(l)\,dl, \tag{24}$$

where

$$R_{\lambda-2}(v,l) = \begin{cases} -\dfrac{(v-l)^{\lambda-2}}{(\lambda-2)(\beta_2-\beta_1)} + \dfrac{v-\beta_1}{\beta_2-\beta_1}(v-l)^{\lambda-3}, & \beta_1 \le l \le v; \\ -\dfrac{(v-l)^{\lambda-2}}{(\lambda-2)(\beta_2-\beta_1)} + \dfrac{v-\beta_2}{\beta_2-\beta_1}(v-l)^{\lambda-3}, & v \le l \le \beta_2. \end{cases}$$

Using $\varphi''(v)$ in (23) and with the use of Fubini's theorem, the identity (22) will be obtained. □

The generalization of (4) for the λ-convex function is provided by the next result.

Theorem 3.2. *Assume* (H_1) *and* (H_2)*, suppose that* $\theta_i(i = 1,2,3)$ *and* $G_k(k = 1,2,3,4)$ *are the same as defined in* $(5)-(7)$ *and* $(11)-(14)$ *respectively, further suppose* φ *is* λ*-convex function, if*

$$\int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right) R_{\lambda-2}(v,l)\,dv \ge 0, \tag{25}$$

for all $v \in [\beta_1,\beta_2]$*, then*

$$\theta_i(\varphi) \ge \sum_{k=0}^{\lambda-4} \frac{1}{k!\,(k+2)(\beta_2-\beta_1)} \int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right)\left[\varphi^{(k+3)}(\beta_1)(v-\beta_1)^{k+2} - \varphi^{(k+3)}(\beta_2)(\right. + \frac{\varphi'(\beta_2)-\varphi'(\beta_1)}{\beta_2-\beta_1} \int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right) dv. \tag{26}$$

Proof. Since φ is an λ-convex function, $\varphi^{(\lambda)} \ge 0$. By Theorem 1.41 given in [1], the function $\varphi^{(k)}$ exists and $(\lambda-k)$-convex for all $k \in 1,\dots,\lambda-2$. On applying Theorem 3.1, we get (26). □

Theorem 3.3. *Assume* (H_1) *and* (H_2)*, let* $\varphi:[\beta_1,\beta_2] \to \mathbb{R}$ *be a function, if* φ *is* λ*-convex function then the following two results are valid.*
(i) *If* $\lambda \ge 3$ *is an odd integer, then* (26) *holds.*
(ii) *Suppose* (26) *be satisfied, if*

$$T(v) := \sum_{k=0}^{\lambda-4} \frac{1}{k!\,(k+2)(\beta_2-\beta_1)}\left[\varphi^{(k+3)}(\beta_1)(v-\beta_1)^{k+2} - \varphi^{(k+3)}(\beta_2)(v-\beta_2)^{k+2}\right] + \frac{\varphi'(\beta_2)-\varphi'(\beta_1)}{\beta_2-\beta_1} \geq 0, \tag{27}$$

then

$$\theta_i(\varphi) \geq 0. \tag{28}$$

Proof. **(i)** For $\lambda \geq 3$ odd integers the function $R_{\lambda-2}$ is positive. Also from Remark A, $\theta_i\big(G_k(.,v)\big) \geq 0$, therefore (25) holds. Hence by using Theorem 3.2, we get (26).
(ii) Using the positivity of $\theta_i\big(G_k(.,v)\big)$ and (27), we get (28). □

4. Information Theory

In this section, the new generalized results are used to estimate different divergences and entropies like; Csiszár divergence, Shannon entropy, Kullback-Leibler divergence, and Zipf-Mandelbrot law.
Csiszár [19, 20] is defined as follows:
Suppose $\psi: (0,\infty) \to \mathbb{R}$ is a positive function. Let $r := (r_1, \dots, r_n)$ and $s := (s_1, \dots, s_n)$ be positive probability distributions. The Csiszár f-divergence $I_\psi(r,s)$ is defined as

$$I_\psi(r,s) := \sum_{i=1}^{n} s_i\, \psi\left(\frac{r_i}{s_i}\right). \tag{29}$$

L. Horv´ath, et al. [21] generalized the Csiszár divergence as follows.
Suppose $\psi: (0,\infty) \to \mathbb{R}$ be a function. Let $r := (r_1, \dots, r_n)$ be a real n-tuple and $s := (s_1, \dots, s_n)$ be a positive n-tuple such that

$$\frac{r_j}{s_j} \in I,\ j = 1, \dots, n. \tag{30}$$

Then let

$$\hat{I}_\psi(r,s) := \sum_{j=1}^{n} s_j\, \psi\left(\frac{r_j}{s_j}\right). \tag{31}$$

Theorem 4.1. *Assume* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$, *let* $\mathbf{r} = (r_1, \dots, r_n)$ *and* $\mathbf{s} = (s_1, \dots, s_n)$ *are n-tuples lies in* $(0,\infty)^n$, *suppose* I *is an interval, with*

$$\frac{r_k}{s_k} \in I,\ k = 1, \dots, n.$$

If $\varphi: (0,\infty) \to \mathbb{R}$ *is an* λ*-convex function, such that* $\varphi \in C^{\lambda}[\beta_1, \beta_2]$ *(where* $\lambda \geq 4$ *is supposed to be an odd integer), then for* $i = 1,2,3$,

$$\theta_{i,cis}(\varphi) \geq \sum_{k=0}^{\lambda-4} \frac{1}{k!\,(k+2)(\beta_2-\beta_1)} \int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right)\left[\varphi^{(k+3)}(\beta_1)(v-\beta_1)^{k+2} - \varphi^{(k+3)}(\beta \right. \\ \left. + \frac{\varphi'(\beta_2)-\varphi'(\beta_1)}{\beta_2-\beta_1} \int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right) dv. \tag{32}$$

where

$$\theta_{1,cis}(\varphi) = \sum_{i=1}^{m}\left(\sum_{j\in N_i} q_j\right)\left[\frac{M_i-\bar{z}_i}{M_i-m_i}\varphi(m_i) + \frac{\bar{z}_i-m_i}{M_i-m_i}\varphi(M_i)\right] - \sum_{k=1}^{n} s_k\, \varphi\left(\frac{r_k}{s_k}\right), \tag{33}$$

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$$\theta_{2,cis}(\varphi) = (Q_n S_n)\left[\frac{M-\bar{z}}{M-m}\varphi(m) + \frac{\bar{z}-m}{M-m}\varphi(M)\right] - \sum_{i=1}^{m}\left(\sum_{j\in N_i} p_j\right)\left[\frac{M_i-\bar{z}_i}{M_i-m_i}\varphi(m_i) + \frac{\bar{z}_i-m_i}{M_i-m_i}\varphi(M_i)\right], \tag{34}$$

and

$$\theta_{3,cis}(\varphi) = -(Q_n S_n)\left[\frac{M-\bar{z}}{M-m}\varphi(m) + \frac{\bar{z}-m}{M-m}\varphi(M)\right] + \sum_{k=1}^{n} s_k\,\varphi\left(\frac{r_k}{s_k}\right). \tag{35}$$

Proof. Since the functions $G_k(.,v)$ are convex for $k = 1,2,3,4$, therefore $\theta_i\big(G_k(.,v)\big) \geq 0$. Also note that $R_{\lambda-2}$ is positive for every odd number λ greater than 2, so (25) holds. Consider $q_k = \frac{s_k}{\sum_{k=1}^{n} s_k}$ and $z_k = \frac{r_k}{s_k}$ in (26) and multiplying by $-Q_n S_n$, we get (32). □

Remark 4.2. *For $i = 3$ the inequality* (32) *gives*

$$\hat{I}_\varphi(r,s) \geq \sum_{k=0}^{\lambda-4} \frac{1}{k!\,(k+2)(\beta_2-\beta_1)}\int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right)\left[\varphi^{(k+3)}(\beta_1)(v-\beta_1)^{k+2} - \varphi^{(k+3)}(\beta_2 + \frac{\varphi'(\beta_2)-\varphi'(\beta_1)}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right)dv + (Q_n S_n)\left[\frac{M-\bar{z}}{M-m}\varphi(m) + \frac{\bar{z}-m}{M-m}\varphi(M\right.\right. \tag{36}$$

Proof. Since

$$\hat{I}_\varphi(r,s) = \sum_{k=1}^{n} s_k\,\varphi\left(\frac{r_k}{s_k}\right), \tag{37}$$

therefore for $i = 3$ in the inequality (32) gives we have (36). □

4.1. **Inequalities for Shannon Entropy.** For a positive probability distribution $r = (r_1, \dots, r_n)$, the Shannon entropy S [21] is defined as

$$S := -\sum_{s=1}^{n} r_s \ln(r_s). \tag{38}$$

Corollary 4.3. *Assume* (H_1) *and* (H_2). *If $r = (r_1, \dots, r_n)$, then*

$$S \leq \sum_{k=0}^{\lambda-4} \frac{(-1)^{(k+2)}(k+1)}{(\beta_2-\beta_1)}\int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right)\left[\frac{(v-\beta_1)^{k+2}}{(\beta_1)^{k+3}} - \frac{(v-\beta_2)^{k+2}}{(\beta_2)^{k+3}}\right]dv - \frac{1}{\beta_1\beta_2}\int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right)dv + (Q_n S_n)\left[\frac{M-\bar{z}}{M-m}\ln(m) + \frac{\bar{z}-m}{M-m}\ln(M)\right]. \tag{39}$$

Proof. Using $\varphi(x) = -\ln(x)$ in (36), we get

$$-\sum_{k=1}^{n} s_k \ln\left(\frac{r_k}{s_k}\right)$$

$$\geq -\sum_{k=0}^{\lambda-4} \frac{1}{k!\,(k+2)(\beta_2-\beta_1)} \int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right)\left[\ln^{(k+3)}(\beta_1)(v-\beta_1)^{k+2} - \ln\right. \tag{40}$$

$$\left. + \frac{1}{\beta_2\beta_1} \int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right) dv - (Q_n S_n)\left[\frac{M-\bar{z}}{M-m}\ln(m) + \frac{\bar{z}-m}{M-m}\ln(M)\right]\right.$$

Since for $\varphi(x) = \ln(x)$, the $(k+3) - th$ derivative is given as

$$\varphi^{(k+3)}(x) = (-1)^{(k+2)} \frac{(k+2)!}{x^{k+3}}. \tag{41}$$

Use (41) in (40) and multiply by -1 on both sides, we get

$$-\sum_{k=1}^{n} s_k \ln\left(\frac{s_k}{r_k}\right) \leq \sum_{k=0}^{\lambda-4} \frac{(-1)^{k+2}(k+2)!}{k!\,(k+2)(\beta_2-\beta_1)} \int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right)\left[\frac{(v-\beta_1)^{k+2}}{(\beta_1)^{k+3}} - \frac{(v-\beta_2)^{k+2}}{(\beta_2)^{k+3}}\right.$$

$$\left. - \frac{1}{\beta_2\beta_1} \int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right) dv + (Q_n S_n)\left[\frac{M-\bar{z}}{M-m}\ln(m) + \frac{\bar{z}-m}{M-m}\ln(M)\right]\right. \tag{42}$$

Taking $r = (1, \dots, 1)$ and using the properties of ln an (42), and after simplification, it become (39). □

4.2. **Inequalities for Kullback-Leibler divergence.** For two positive probability distributions $\mathbf{r} = (r_1, \dots, r_n)$ and $\mathbf{s} = (s_1, \dots, s_n)$, the Kullback-Leibler divergence [21] is denoted by $D(r,s)$ and defined by

$$D(r,s) := \sum_{j=1}^{n} r_j \ln\left(\frac{r_j}{s_j}\right). \tag{43}$$

Corollary 4.4. *Assume* (H_1) *and* (H_2). *If* $r = (r_1, \dots, r_n)$ *and* $s = (s_1, \dots, s_n)$, *then*

$$D(s,r) \geq \sum_{k=0}^{\lambda-4} \frac{(-1)^{(k+3)}(k+1)}{\beta_2-\beta_1} \int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right)\left[\frac{(v-\beta_1)^{k+2}}{(\beta_1)^{k+3}} - \frac{(v-\beta_2)^{k+2}}{(\beta_2)^{k+3}}\right] dv$$

$$+ \frac{1}{\beta_2\beta_1} \int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right) dv - (Q_n S_n)\left[\frac{M-\bar{z}}{M-m}\ln(m) + \frac{\bar{z}-m}{M-m}\ln(M)\right]. \tag{44}$$

Proof. Using $\varphi(x) = -\ln(x)$ in (36), we get

$$-\sum_{k=1}^{n} s_k \ln\left(\frac{r_k}{s_k}\right) \geq -\sum_{k=0}^{\lambda-4} \frac{1}{k!(k+2)(\beta_2-\beta_1)} \int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right)[\ln^{(k+3)}(\beta_1)(v-\beta_1)^{k+2}$$

$$-\ln^{(k+3)}(\beta_2)(v-\beta_2)^{k+2}]dv + \frac{1}{\beta_2\beta_1} \int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right) dv - (Q_n S_n)\left[\frac{M-\bar{z}}{M-m}\ln(m) + \frac{\bar{z}-m}{M-m}\ln(M)\right].$$

Using the property of ln which is $-\ln x = \ln(x)^{-1} = \ln\left(\frac{1}{x}\right)$, we get

$$\sum_{k=1}^{n} s_k \ln\left(\frac{s_k}{r_k}\right) \geq -\sum_{k=0}^{\lambda-4} \frac{1}{k!(k+2)(\beta_2-\beta_1)} \int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right)[\ln^{(k+3)}(\beta_1)(v-\beta_1)^{k+2}$$

$$-\ln^{(k+3)}(\beta_2)(v-\beta_2)^{k+2}]dv + \frac{1}{\beta_2\beta_1} \int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right) dv - (Q_n S_n)\left[\frac{M-}{M-}\right. \tag{45}$$

Since for $\varphi(x) = \ln(x)$, the $(k+3)$-th derivative is given as $\varphi^{(k+3)}(x) = (-1)^{(k+2)} \frac{(k+2)!}{x^{k+3}}$ Use the properties of ln on the larger side of the inequality (45),

$$\sum_{k=1}^{n} s_k \ln\left(\frac{s_k}{r_k}\right) \geq \sum_{k=0}^{\lambda-4} \frac{(-1)^{(k+3)}(k+2)!}{k!\,(k+2)(\beta_2-\beta_1)} \int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right)\left[\frac{(v-\beta_1)^{k+2}}{(\beta_1)^{k+3}} - \frac{(v-\beta_2)^{k+2}}{(\beta_2)^{k+3}}\right] dv + \frac{1}{\beta_2\beta_1}\int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right) dv - (Q_n S_n)\left[\frac{M-\bar{z}}{M-m}\ln(m) + \frac{\bar{z}-m}{M-m}\ln(M)\right]. \quad (46)$$

Hence, we get (44). □

4.3. **Inequalities for Zipf-Mandelbrot Law.** The Zipf-Mandelbrot entropy for positive integer n, and for real numbers $r \in [0,\infty), s \in (0,\infty)$ is defined as

$$Z(H,r,s) = \frac{s}{H_{r,s}^n}\sum_{\kappa=1}^{n}\frac{\ln(\kappa+r)}{(\kappa+r)^s} + \ln\left(H_{r,s}^n\right), \quad (47)$$

where

$$H_{r,s}^n = \sum_{\upsilon=1}^{n}\frac{1}{(\upsilon+r)^s}. \quad (48)$$

Consider the following function

$$r_\kappa = \phi(\kappa;n,r,s) = \frac{1}{(\kappa+r)^s H_{r,s}^n}. \quad (49)$$

Corollary 4.5. *Assume* **(H₁)** *and* **(H₂)**. *If* $r = (r_1,\dots,r_n)$ *and* $s = (s_1,\dots,s_n)$, *then*

$$S = Z(H,r,s) \leq \sum_{k=0}^{\lambda-4} \frac{(-1)^{(k+2)}(k+1)}{\beta_2-\beta_1} \int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right)\left[\frac{(v-\beta_1)^{k+2}}{(\beta_1)^{k+3}} - \frac{(v-\beta_2)^{k+2}}{(\beta_2)^{k+3}}\right] dv - \frac{1}{\beta_1\beta_2}\int_{\beta_1}^{\beta_2} \theta_i\left(G_k(.,v)\right) dv + (Q_n S_n)\left[\frac{M-\bar{z}}{M-m}\ln(m) + \frac{\bar{z}-m}{M-m}\ln(M)\right]. \quad (50)$$

Proof. Note that $\sum_{\kappa=1}^{n} r_\kappa = \sum_{\kappa=1}^{n}\frac{1}{(\kappa+r)^s H_{r,s}^n} = \frac{H_{r,s}^n}{H_{r,s}^n} = 1$, therefore use r_κ in Shannon entropy give Mandelbrot entropy that is:

$$S_r = -\sum_{\kappa=1}^{n} r_\kappa \ln(r_\kappa) = Z(H,r,s). \quad (51)$$

Thus using r_κ in (39), gives (50). □

Corollary 4.6. *Assume* **(H₁)**. *For* $r_1, r_2 \geq 0$ *and* $s_1, s_2 \geq 0$, *suppose* $H_{r_1,s_1}^n = \sum_{\upsilon=1}^{n}\frac{1}{(\upsilon+r_1)^{s_1}}$ *and* $H_{r_2,s_2}^n = \sum_{\upsilon=1}^{n}\frac{1}{(\upsilon+r_2)^{s_2}}$. *If* $r_\kappa = \frac{1}{(\kappa+r_1)^{s_1}H_{r_1,s_1}^n}$ *and* $s_\kappa = \frac{1}{(\kappa+r_2)^{s_2}H_{r_2,s_2}^n}$, *then*

$$D(r,s) = \left[\sum_{\kappa=1}^{n}\frac{1}{(\kappa+r_1)^{s_1}H_{r_1,s_1}^n}\ln\frac{(\kappa+r_2)^{s_2}H_{r_2,s_2}^n}{(\kappa+r_1)^{s_1}H_{r_1,s_1}^n}\right] = -Z(H,r_2,s_2) + \frac{s_1}{H_{r_2,s_2}^n}\sum_{\kappa=1}^{n}\frac{\ln(\kappa+r_1}{(\kappa+r_2)^{s}}$$
$$+\ln\left(H_{r_1,s_1}^n\right) \geq \sum_{k=0}^{\lambda-4}\frac{(-1)^{(k+3)}(k+1)}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2}\theta_i\left(G_k(.,v)\right)\left[\frac{(v-\beta_1)^{k+2}}{(\beta_1)^{k+3}} - \frac{(v-\beta_2)}{(\beta_2)^{k}}\right. \quad (52)$$
$$\left. + \frac{1}{\beta_2\beta_1}\int_{\beta_1}^{\beta_2}\theta_i\left(G_k(.,v)\right) dv - (Q_n S_n)\left[\frac{M-\bar{z}}{M-m}\ln(m) + \frac{\bar{z}-m}{M-m}\ln(M)\right]\right.$$

Proof. Using $r_\kappa = \frac{1}{(\kappa+r_1)^{s_1}H_{r_1,s_1}^n}$ and $s_\kappa = \frac{1}{(\kappa+r_2)^{s_2}H_{r_2,s_2}^n}$, in (44) gives (52). □

5. CONCLUSION

The inequality given by Pečarić *et al.* [17] is generalized for higher-order convex functions using two interpolations Abel-Gontscharoff and Montgomery identities at a time. New identities are constructed using the non-negative difference of the inequality given in [17] and Abel-Gontscharoff and Montgomery identities. The higher order convexity is discussed for different cases of the domains. These generalizations are a new addition to the literature to study higher-order convexity. The generalized inequality is used to estimate the entropies of information theory like; Csiszár divergence, Shannon entropy, Kullback-Leibler divergence, and Zipf-Mandelbrot law for higher order convex function. Further we can also investigate the bounds for the identities related to the obtained generalized inequality using inequalities using Čebyšev functional and results relating to the Grüss and Ostrowski type inequalities can be constructed as given in Section 3 of [8]. Also we can construct the non-negative functionals from inequalities (26) and give related mean value theorems and we can construct the new families of m-exponentially convex functions and Cauchy means related to these functionals as given in Section 4 of [8].

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