

ON EXACT SOLITON SOLUTIONS OF NONLINEAR SPACE-TIME FRACTIONAL KG & ZK-BBM EQUATIONS ARISING IN WAVE PHENOMENA VIA (G'/G) -EXPANSION METHOD

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Abstract

In this study, the multiple closed-form solutions of nonlinear space-time fractional partial differential equations arising in wave phenomena are obtained. The well known $(\frac{G'}{G})$ -expansion technique is used to find the travelling wave solutions of well-known equations namely like Klein-Gordon equation (KGE) and Zakharov-Kuznetsov Benjamin Bona Mahoney (ZK-BBM) equation are achieved in sense of hyperbolic function, Rational function and Trigonometric functions. The nonlinear space-time FDE equations are first transform into nonlinear ODE by using transformations. After using $(\frac{G'}{G})$ -expansion the NODE transforms into nonlinear system that is solved for required parameters by using Matlab. The obtained solutions are entirely new. It will be beneficial to comprehend the physical behavior of the equations. Finally, the graphical illustrations of the solutions are also presented for different values with the help of MATLAB.

1. Introduction

FPDE are generalization of formal differential equations, involve the non-integer order integration and differentiation and represent the generality of the formal integral and differential calculus. The applications of integral equations and FDE's have been concentrated a great studies caused by their regularly uses in various areas of engineering and science like signal processing, biological mathematics, system verification, control theory, finance, electrochemistry, viscoelastic material and fractional diffusion and has captivated more attention of many researchers. In recently years, the sufficient works have been done to obtained exact and numeric solutions

of ODE and FDE [1-3], of real concern [4-6]. Newly, numerous sophisticated methodologies have been proposed to attain both the precise and numerical solutions of fractional partial differential equations (FPDEs). [7-10] like Modified simple equation method [11-12], Fractional transform method [13-14], q-Homopy analysis transform method [15-16], Shooting method [17], Integral transform [18-19], Homotopy perturbation approach [20-21], Modified simple method [22-23], Variational iteration method [24-25], The fractional sub-equation method [26-27], Residual power series method [28-29], The sine-cosine approach [30], The standard tanh and extended tanh methods[31-32],The exp-function method [33-

34], Hirota bilinear methods [35-36], The Cole-Hopf transformation method [37], The direct algebraic method [38], The generalized Riccati equation method [39], Fractional bell collocation method [40] and so on.

In this article, our aim is to determine more general exact soliton solutions of two NSTFDE's, namely the KGE and ZK-BBM equation on the basis of accordant non-integer derivative.

The first model taken into consideration is KGE, a basic nonlinear equation that appears when the relativistic wave equation in quantum physics is analyzed. This equation is a vital tool in many areas of physics and is widely used in both classical and quantum field theory [41]. This model is provided by

$$D_t^{2\beta} u + D_x^{2\alpha} u + \gamma u - \delta u^2 = 0; \quad 0 < \alpha, \beta \leq 1, t > 0 \quad (1)$$

The ZK-BBM equation, which explains the gravity of water waves, is the second model examined. This describes how long waves propagate in a particular nonlinear dispersion system in a single direction, especially in the long wave regime. The (ZK-BBM) equation has been used to explain several physical phenomena, including shock waves and solitary

wave interaction [42]. The model (ZK-BBM) is provided by

$$D_t^\beta u + D_x^\alpha u - 2au(D_x^\alpha u) - bD_t^\beta(D_x^{2\alpha} u) = 0 \quad (2)$$

In both model γ, β, a , and b are random parameters and α is an order of the FDE's.

This study orderly as under: in Section 2, we represent some common properties of the modified Riemann-Liouville derivatives and the steps of $\left(\frac{G'}{G}\right)$ -expansion approach is provided in Section 3, in Section 4, through the mentioned method in Section 3, examine closed-form solutions of KGE and ZK-BBM in Section 5. Physically descriptions of some results are expressed and also remarks are point out in Section 6.

2. Derivative operator and its Properties

This study describes $\left(\frac{G'}{G}\right)$ -expansion method to achieve exact soliton solutions involving time and space fractional order derivatives in nonlinear differential equations with the modified Riemann-Liouville derivative and β is order defined by [43]

$$D_t^\beta f(t) = \begin{cases} \frac{1}{\Gamma(1-\beta)} \int_0^t (t-\zeta)^{-\beta-1} [g(\zeta) - g(0)] d\zeta, & \text{if } \beta < 0 \\ \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-\zeta)^{-\beta} [g(\zeta) - g(0)] d\zeta, & \text{if } 0 < \beta < 1 \\ [g^{(\beta-n)}(t)]^{(n)}, & \text{if } n \leq \beta < n+1, n \geq 1 \end{cases} \quad (3)$$

$\Gamma(*)$ is the Gamma function which is defined by:

$$\Gamma(\beta) = \lim_{n \rightarrow \infty} \frac{n! n^\beta}{(\beta+1)(\beta+2) \dots (\beta+n)} \quad (4)$$

Properties

$$i) \quad D_t^\beta t^r = \frac{\Gamma(1+r)}{\Gamma(r-\beta+1)} t^{r-\beta} \quad (5)$$

$$ii) \quad D_t^\beta [g(t)h(t)] = h(t)D_t^\beta g(t) + g(t)D_t^\beta h(t) \quad (6)$$

$$iii) \quad D_t^\beta g[h(t)] = \frac{dg(h(t))}{dh(t)} D_t^\beta h(t) \quad (7)$$

3. Methodology

In way to understand the $\left(\frac{G'}{G}\right)$ -Expansion method, suppose the nonlinear STFDE in two independent parameters x and t in the format

$$Q(u, u_x, u_t, u_{xx}, \dots, D_t^\beta u, D_x^\alpha u, \dots) = 0; \quad 0 < a \leq 1 \quad (8)$$

Where $D_t^\beta u, D_x^\alpha u$ are fractional derivatives operator of u , $u(x, t)$ is unknown function, Q is a polynomial in u , and its various fractional derivatives and nonlinear terms are concerned with linear terms:

Step 1: Consider travelling wave transformation as follows

$$u(x, t) = u(\zeta), \text{ where } \zeta = \frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\beta}{\Gamma(1+\beta)} \quad (9)$$

Where $k, c \neq 0$ parameters to be determined later. The soliton wave variable Eq. (9) allows Eq. (8) into nonlinear ODE for $G = G(\zeta)$ as,

$$Q(u, ku', -cu', k^2u'', \dots, -cu', \dots) = 0, \quad (10)$$

Step 2: Integrating Eq. (10) if it contains derivatives in each term and assume constant(s) of integration is zero.

Step 3: Assume the solution of nonlinear ODE given in Eq. (10) can be written through a polynomial in $\left(\frac{G'}{G}\right)$ as follow

$$u(\zeta) = \sum_{k=0}^m a_k (G'/G)^k, \quad a_m \neq 0 \quad (11)$$

For $a_k (k = 0, 1, 2, \dots, m)$ are constant(s) to determine latter although $G = G(\zeta)$ will satisfy the second order LODE

$$G''(\zeta) + \lambda G'(\zeta) + vG(\zeta) = 0 \quad (12)$$

Where $G' = \frac{dG}{d\zeta}$, $G'' = \frac{d^2G}{d\zeta^2}$, λ and v are random parameters to determine later. Now, non integer m can be found by assuming a homogeneity between nonlinear terms and largest order derivative occurring in Eq. (10). To the letter, we describe the degree of $u(\zeta)$ like $\text{Deg}[u(\zeta)] = m$ that leads to the degree of the other expression such as [43]

$$\text{Deg} \left[\frac{d^p u}{d\zeta^p} \right] = m + p \quad (13)$$

$$\text{Deg} \left[U^s \left(\frac{d^p u}{d\zeta^p} \right)^q \right] = ms + q(p + m) \quad (14)$$

By the generalized solution of Eq. (12), we get [44]

$$\left(\frac{G'}{G} \right) = \begin{cases} -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4v}}{2} \left(\frac{c_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4v}}{2} \zeta\right) + c_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4v}}{2} \zeta\right)}{c_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4v}}{2} \zeta\right) + c_2 \sinh\left(\frac{\sqrt{\lambda^2 - 4v}}{2} \zeta\right)} \right), & \lambda^2 - 4v > 0 \\ -\frac{\lambda}{2} + \frac{\sqrt{4v - \lambda^2}}{2} \left(\frac{-c_1 \sin\left(\frac{\sqrt{4v - \lambda^2}}{2} \zeta\right) + c_2 \cos\left(\frac{\sqrt{4v - \lambda^2}}{2} \zeta\right)}{c_1 \cos\left(\frac{\sqrt{4v - \lambda^2}}{2} \zeta\right) + c_2 \sin\left(\frac{\sqrt{4v - \lambda^2}}{2} \zeta\right)} \right), & \lambda^2 - 4v < 0 \\ -\frac{\lambda}{2} + \frac{c_2}{c_1 + c_2 \zeta}, & \lambda^2 - 4v = 0 \end{cases} \quad (15)$$

c_1 and c_2 are arbitrary parameters.

Step 4: Plugging Eq. (11) with Eq. (12) into the Eq. (10), combining all terms of identical powers $\left(\frac{G'}{G}\right)$, then Eq. (10) is changed within another algebraic expression in $\left(\frac{G'}{G}\right)$, equate coefficient of each expression to zero and a system is obtained for $a_k (k = 0, 1, 2, \dots, m)$, λ, k, c and v will be obtained.

Step 5: from algebraic expression in step 4, we can find constants $a_k (k = 0, 1, 2, \dots, m)$, λ, k, c and v with the help of Matlab. Since the generalized solution of LODE (12) have been well-known for us, then plugging $a_k (k =$

$0, 1, 2, \dots, m), k$ and c into generalized solutions of Eq. (12) into Eq. (11), we achieve a series of fundamental solutions of the nonlinear evaluation Eq. (8).

4. Applications of $\left(\frac{G'}{G}\right)$ -Expansion technique

The aim of this section is to achieve the exact soliton solutions of nonlinear STFDE's given in Eq. (1) and Eq. (2) through the $\left(\frac{G'}{G}\right)$ -expansion technique.

4.1 Fractional Klein-Gardner equation (FKGE)

We consider soliton wave variable of Eq.(1) and apply the transformation $u(x, t) = u(\zeta)$ and $\zeta = \frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\beta}{\Gamma(1+\beta)}$, Where $k, c \neq 0$ constants to be determined latter and the fractional parameter α, β lies $0 < \alpha, \beta \leq 1$. By plugging Eq. (9) in Eq. (1), the equation can be reduce to nonlinear ODE.

$$c^2 u'' + k^2 u'' + \gamma u - \delta u^2 = 0 \quad (16)$$

Suppose the solution of ODE Eq. (16) is Eq. (11). By using Eq. (13) and Eq. (14) the homogenous balance between

$$u^2 \text{ and } u'' \text{ is given as } 2 + m = 2m \Rightarrow m = 2$$

Thus from Eq. (11), we have

$$u(\zeta) = a_0 + a_1 \left(\frac{G'}{G}\right) + a_2 \left(\frac{G'}{G}\right)^2 \quad a_2 \neq 0 \quad (17)$$

$$u^2 = a_0^2 + a_1^2 \left(\frac{G'}{G}\right)^2 + a_2^2 \left(\frac{G'}{G}\right)^4 + 2a_0 a_1 \left(\frac{G'}{G}\right) + 2a_1 a_2 \left(\frac{G'}{G}\right)^3 + 2a_0 a_2 \left(\frac{G'}{G}\right)^2 \quad (18)$$

By using Eq. (12) and Eq. (17), we get

$$u'' = 6a_2 \left(\frac{G'}{G}\right)^4 + (10a_2 \lambda + 2a_1) \left(\frac{G'}{G}\right)^3 + (3a_1 \lambda + 8a_2 v + 4a_2 \lambda^2) \left(\frac{G'}{G}\right)^2 + (2a_1 v + 6a_2 \lambda v + a_1 \lambda^2) \left(\frac{G'}{G}\right) + a_1 \lambda v + 2a_2 v^2 \quad (19)$$

By plugging Eq. (17)-Eq. (19) into Eq. (16) and by equate coefficients of identical powers of $\left(\frac{G'}{G}\right)$ and get the algebraic system for a_0, a_1, a_2, k and c as follows:

$$\begin{aligned} \left(\frac{G'}{G}\right)^0 : a_1 \lambda v c^2 + 2a_2 v^2 c^2 + a_1 \lambda v k^2 + 2a_2 v^2 k^2 + a_0 \gamma - \delta a_0^2 &= 0 \\ \left(\frac{G'}{G}\right)^1 : 6a_2 \lambda v c^2 + 2a_1 v c^2 + a_1 \lambda^2 c^2 + 6a_2 \lambda v k^2 + 2a_1 v k^2 + a_1 \lambda^2 k^2 + a_1 \gamma - 2a_0 a_1 \delta &= 0 \\ \left(\frac{G'}{G}\right)^2 : 8a_2 v c^2 + 3a_1 \lambda c^2 + 4a_2 \lambda^2 c^2 + 8a_2 v k^2 + 3a_1 \lambda k^2 + 4a_2 \lambda^2 k^2 + a_2 \gamma - a_1^2 \delta - &2a_0 a_2 \delta = 0 \\ \left(\frac{G'}{G}\right)^3 : 2a_1 c^2 + 10a_2 \lambda c^2 + 2a_1 k^2 + 10a_2 \lambda k^2 - 2a_1 a_2 \delta &= 0 \\ \left(\frac{G'}{G}\right)^4 : 6a_1 c^2 + 6a_2 k^2 - a_2^2 \delta &= 0 \end{aligned}$$

After solving the above algebraic system, the constants a_0, a_1, a_2, c and k can be achieved with the help of MATLAB, we have one set of solution

$$K = 1, a_0 = \frac{6\gamma v}{(4v - \lambda^2)\delta}, a_1 = \frac{6\lambda\gamma}{(4v - \lambda^2)\delta}, a_2 = \frac{6\gamma}{(4v - \lambda^2)\delta}, c = \frac{\sqrt{\gamma \cdot 4v + \lambda^2}}{\sqrt{4v \cdot \lambda^2}} \quad (20)$$

Where λ, γ, δ and v are constant parameters, by using Eq. (20), the equation Eq. (17) can be expressed as

$$u(\zeta) = \frac{6\gamma v}{(4v - \lambda^2)\delta} + \frac{6\lambda\gamma}{(4v - \lambda^2)\delta} \left(\frac{G'}{G}\right) + \frac{6\gamma}{(4v - \lambda^2)\delta} \left(\frac{G'}{G}\right)^2 \quad (21)$$

when $\zeta = \frac{x^\alpha}{\Gamma(1+\alpha)} - \left(\frac{\sqrt{\gamma \cdot 4v + \lambda^2}}{\sqrt{4v \cdot \lambda^2}}\right) \frac{t^\beta}{\Gamma(1+\beta)}$.

Plugging the generalize solutions Eq. (12) into Eq. (21), we get different types of soliton solution of the Eq. (1)

Case I: If $-4v + \lambda^2 > 0$

$$u(\zeta) = \frac{6\gamma v}{(4v-\lambda^2)\delta} + \frac{6\lambda\gamma}{(4v-\lambda^2)\delta} \left(-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4v}}{2} \left(\frac{c_1 \sinh\left(\frac{\sqrt{\lambda^2-4v}}{2}\zeta\right) + c_2 \cosh\left(\frac{\sqrt{\lambda^2-4v}}{2}\zeta\right)}{c_1 \cosh\left(\frac{\sqrt{\lambda^2-4v}}{2}\zeta\right) + c_2 \sinh\left(\frac{\sqrt{\lambda^2-4v}}{2}\zeta\right)} \right) \right) + \frac{6\gamma}{(4v-\lambda^2)\delta} \left(-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4v}}{2} \left(\frac{c_1 \sinh\left(\frac{\sqrt{\lambda^2-4v}}{2}\zeta\right) + c_2 \cosh\left(\frac{\sqrt{\lambda^2-4v}}{2}\zeta\right)}{c_1 \cosh\left(\frac{\sqrt{\lambda^2-4v}}{2}\zeta\right) + c_2 \sinh\left(\frac{\sqrt{\lambda^2-4v}}{2}\zeta\right)} \right) \right)^2$$

Case II: If $-4v + \lambda^2 < 0$

$$u(\zeta) = \frac{6\gamma v}{(4v-\lambda^2)\delta} + \frac{6\lambda\gamma}{(4v-\lambda^2)\delta} \left(-\frac{\lambda}{2} + \frac{\sqrt{4v-\lambda^2}}{2} \left(\frac{-c_1 \sin\left(\frac{\sqrt{4v-\lambda^2}}{2}\zeta\right) + c_2 \cos\left(\frac{\sqrt{4v-\lambda^2}}{2}\zeta\right)}{c_1 \cos\left(\frac{\sqrt{4v-\lambda^2}}{2}\zeta\right) + c_2 \sin\left(\frac{\sqrt{4v-\lambda^2}}{2}\zeta\right)} \right) \right) + \frac{6\gamma}{(4v-\lambda^2)\delta} \left(-\frac{\lambda}{2} + \frac{\sqrt{4v-\lambda^2}}{2} \left(\frac{-c_1 \sin\left(\frac{\sqrt{4v-\lambda^2}}{2}\zeta\right) + c_2 \cos\left(\frac{\sqrt{4v-\lambda^2}}{2}\zeta\right)}{c_1 \cos\left(\frac{\sqrt{4v-\lambda^2}}{2}\zeta\right) + c_2 \sin\left(\frac{\sqrt{4v-\lambda^2}}{2}\zeta\right)} \right) \right)^2$$

Case III: If $-\lambda^2 + 4v = 0$

$$u(\zeta) = \frac{6\gamma v}{(-\lambda^2 + 4v)\delta} + \frac{6\lambda\gamma}{(4v-\lambda^2)\delta} \left(-\frac{\lambda}{2} + \frac{c_2}{c_1 + c_2\zeta} \right) + \frac{6\gamma}{(-\lambda^2 + 4v)\delta} \left(-\frac{\lambda}{2} + \frac{c_2}{c_1 + c_2\zeta} \right)^2$$

where $\zeta = \frac{x^\alpha}{\Gamma(1+\alpha)} - \left(\frac{\sqrt{\gamma-4v+\lambda^2}}{\sqrt{4v-\lambda^2}} \right) \frac{t^\beta}{\Gamma(1+\beta)}$ and c_1 and c_2 are arbitrary constants.

Some Special cases:

Case I: if put $c_1 = 0$, then

$$u_1 = \frac{3\gamma}{2\delta} \left(1 - \coth^2 \left(\frac{\sqrt{\lambda^2-4v}}{2\Gamma(1+\alpha)} x^\alpha - \frac{\sqrt{4v-\gamma-\lambda^2} t^\beta}{2\Gamma(1+\beta)} \right) \right)$$

and if $c_2 = 0$, then

$$u_2 = \frac{3\gamma}{2\delta} \left(1 - \tanh^2 \left(\frac{\sqrt{\lambda^2-4v}}{2\Gamma(1+\alpha)} x^\alpha - \frac{\sqrt{4v-\gamma-\lambda^2} t^\beta}{2\Gamma(1+\beta)} \right) \right)$$

Case II: if put $c_1 = 0$, then

$$u_3 = \frac{3\gamma}{2\delta} \left(1 + \cot^2 \left(\frac{\sqrt{4v-\lambda^2}}{2\Gamma(1+\alpha)} x^\alpha - \frac{\sqrt{\gamma-4v+\lambda^2} t^\beta}{2\Gamma(1+\beta)} \right) \right)$$

and if $c_2 = 0$, then

$$u_4 = \frac{3\gamma}{2\delta} \left(1 + \tan^2 \left(\frac{\sqrt{4v-\lambda^2}}{2\Gamma(1+\alpha)} x^\alpha - \frac{\sqrt{\gamma-4v+\lambda^2} t^\beta}{2\Gamma(1+\beta)} \right) \right)$$

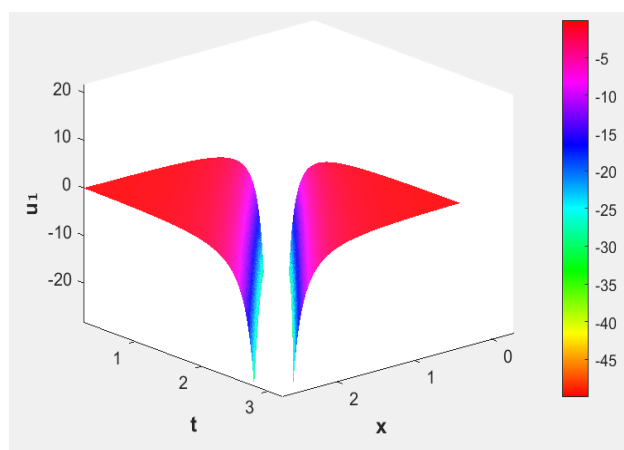


fig1: $\alpha = 0.8, \beta = 0.8, \gamma = 0.6, \lambda = 1.8, v = 1.6, \delta = 1.2$

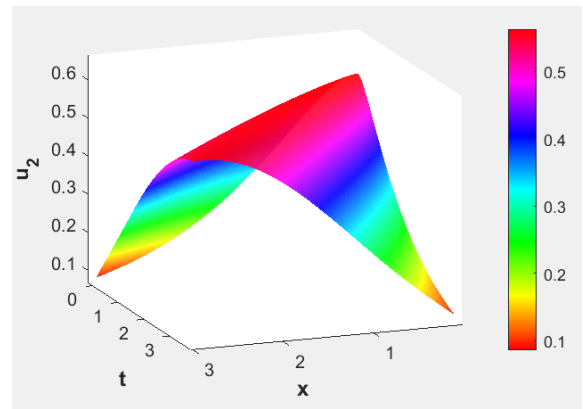


fig2: $\alpha = 0.7, \beta = 0.6, \gamma = 0.5, \lambda = 1.0, v = 0.8, \delta = 0.8$

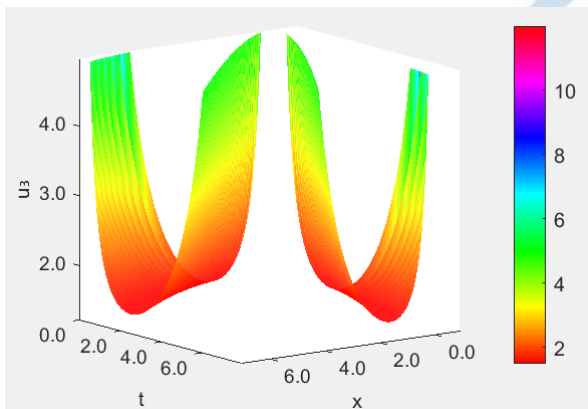


fig3: $\alpha = 0.5, \beta = 0.5, \gamma = 1.0, \lambda = 1.0, v = 1.5, \delta = 1.0$

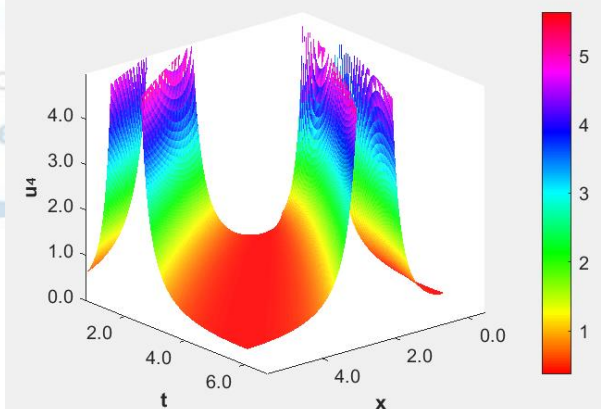


fig4: $\alpha = 0.6, \beta = 0.5, \gamma = 0.5, \lambda = 1.1, v = 3.0, \delta = 2.0$

4.2 Fractional Zakharov-Kuznetsov Benjamin Bona Mahoney Equation (ZK-BBM)

We assume soliton wave variable of Eq. (2) and apply the transformation

$$u(x, t) = u(\zeta) \text{ and } \zeta = \frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\beta}{\Gamma(1+\beta)},$$

Where $k, c \neq 0$ constants and $0 < \alpha, \beta \leq 1$. Putting Eq. (9) in Eq. (2), the equation can be reduce to nonlinear ODE.

$$-cu' + ku' - 2akuu' + bck^2u''' = 0 \quad (22)$$

Integrate Eq. (22) with respect to ζ and suppose integration constant(s) is zero

$$-cu + ku - aku^2 + bck^2u'' = 0 \quad (23)$$

Suppose the solution of ODE Eq. (23) is Eq. (11). By using Eq. (13) and Eq. (14) the homogenous balance between

$$u^2 \text{ and } u'' \text{ is } 2 + m = 2m \Rightarrow m = 2$$

Thus from Eq. (11), we have

$$u(\zeta) = a_0 + a_1 \left(\frac{G'}{G}\right) + a_2 \left(\frac{G'}{G}\right)^2 \quad a_2 \neq 0 \quad (24)$$

$$u^2 = a_0^2 + a_1^2 \left(\frac{G'}{G}\right)^2 + a_2^2 \left(\frac{G'}{G}\right)^4 + 2a_0a_1 \left(\frac{G'}{G}\right) + 2a_1a_2 \left(\frac{G'}{G}\right)^3 + 2a_0a_2 \left(\frac{G'}{G}\right)^2 \quad (25)$$

By using Eq. (12) and Eq. (17), we get

$$u'' = 6a_2 \left(\frac{G'}{G}\right)^4 + (10a_2\lambda + 2a_1) \left(\frac{G'}{G}\right)^3 + (3a_1\lambda + 8a_2v + 4a_2\lambda^2) \left(\frac{G'}{G}\right)^2 + (2a_1v + a_1\lambda^2 + 6a_2\lambda v) \left(\frac{G'}{G}\right) + 2a_2v^2 + a_1\lambda v \quad (26)$$

By plugging Eq. (24)-Eq. (26) into Eq. (23) and by equate coefficients of identical powers of $\left(\frac{G'}{G}\right)$, get the algebraic system for a_0, a_1, a_2, k and c as follows:

$$\begin{aligned} \left(\frac{G'}{G}\right)^0 : ka_0 - ca_0 - aa_0^2k + a_1\lambda vbck^2 + 2a_2v^2 bck^2 &= 0 \\ \left(\frac{G'}{G}\right)^1 : ka_1 - ca_1 - 2aa_0a_1k + 6a_2\lambda v bck^2 + 2a_1vbck^2 + a_1\lambda^2 bck^2 &= 0 \\ \left(\frac{G'}{G}\right)^2 : ka_2 - ca_2 - aa_1^2k - 2aa_0a_2k + 8a_2v bck^2 + 3a_1\lambda bck^2 + 4a_2\lambda^2 bck^2 &= 0 \\ \left(\frac{G'}{G}\right)^3 : -2aa_1a_2k + 2a_1bck^2 + 10a_2\lambda bck^2 &= 0 \\ \left(\frac{G'}{G}\right)^4 : -aa_2^2k + 6a_2bck^2 &= 0 \end{aligned}$$

After solving the above algebraic system, the constants a_0, a_1, a_2, c and k can be achieved with the help of Matlab, we have one set of solution

$$k = 1, a_0 = \frac{6bv}{a(4vb + 1 - b\lambda^2)}, a_1 = \frac{6b\lambda}{a(4vb + 1 - b\lambda^2)}, a_2 = \frac{6b}{a(4vb + 1 - b\lambda^2)}, c = \frac{1}{a(4vb + 1 - b\lambda^2)} \quad (27)$$

Where λ, a, b and v are constant parameters, by using Eq. (26), the equation (23) can be expressed as

$$u(\zeta) = \frac{6bv}{a(4vb + 1 - b\lambda^2)} + \frac{6b\lambda}{a(4vb + 1 - b\lambda^2)} \left(\frac{G'}{G}\right) + \frac{6b}{a(4vb + 1 - b\lambda^2)} \left(\frac{G'}{G}\right)^2 \quad (28)$$

when $\zeta = \frac{x^\alpha}{\Gamma(\alpha+1)} - \left(\frac{1}{a(1+4vb - b\lambda^2)}\right) \frac{t^\beta}{\Gamma(\beta+1)}$

Plugging the generalize solutions Eq. (12) into Eq. (28), we get different types of soliton solutions of the Eq. (2)

Case I: For $\lambda^2 - 4v > 0$

$$u(\zeta) = \frac{6bv}{a(4vb + 1 - b\lambda^2)} + \frac{6b\lambda}{a(4vb + 1 - b\lambda^2)} \left(-\frac{\lambda}{2} \frac{\sqrt{\lambda^2 - 4v}}{2} \left(\frac{c_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4v}}{2} \zeta\right) + c_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4v}}{2} \zeta\right)}{c_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4v}}{2} \zeta\right) + c_2 \sinh\left(\frac{\sqrt{\lambda^2 - 4v}}{2} \zeta\right)} \right) \right) + \frac{6b}{a(4vb + 1 - b\lambda^2)} \left(-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4v}}{2} \left(\frac{c_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4v}}{2} \zeta\right) + c_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4v}}{2} \zeta\right)}{c_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4v}}{2} \zeta\right) + c_2 \sinh\left(\frac{\sqrt{\lambda^2 - 4v}}{2} \zeta\right)} \right) \right)^2$$

Case II: For $\lambda^2 - 4v < 0$

$$u(\zeta) = \frac{6bv}{a(4vb + 1 - b\lambda^2)} + \frac{6b\lambda}{a(4vb + 1 - b\lambda^2)} \left(-\frac{\lambda}{2} + \frac{\sqrt{4v - \lambda^2}}{2} \left(\frac{-c_1 \sin\left(\frac{\sqrt{4v - \lambda^2}}{2} \zeta\right) + c_2 \cos\left(\frac{\sqrt{4v - \lambda^2}}{2} \zeta\right)}{c_1 \cos\left(\frac{\sqrt{4v - \lambda^2}}{2} \zeta\right) + c_2 \sin\left(\frac{\sqrt{4v - \lambda^2}}{2} \zeta\right)} \right) \right) + \frac{6b}{a(4vb + 1 - b\lambda^2)} \left(-\frac{\lambda}{2} + \frac{\sqrt{4v - \lambda^2}}{2} \left(\frac{-c_1 \sin\left(\frac{\sqrt{4v - \lambda^2}}{2} \zeta\right) + c_2 \cos\left(\frac{\sqrt{4v - \lambda^2}}{2} \zeta\right)}{c_1 \cos\left(\frac{\sqrt{4v - \lambda^2}}{2} \zeta\right) + c_2 \sin\left(\frac{\sqrt{4v - \lambda^2}}{2} \zeta\right)} \right) \right)^2$$

Case III: For $\lambda^2 - 4v = 0$

$$u(\zeta) = \frac{6bv}{a(4vb + 1 - b\lambda^2)} + \frac{6b\lambda}{a(4vb + 1 - b\lambda^2)} \left(-\frac{\lambda}{2} + \frac{c_2}{c_1 + c_2 \zeta} \right) \frac{6b}{a(4vb + 1 - b\lambda^2)} \left(-\frac{\lambda}{2} + \frac{c_2}{c_1 + c_2 \zeta} \right)^2$$

where $\zeta = \frac{x^\alpha}{\Gamma(1+\alpha)} - \left(\frac{1}{a(4vb+1-b\lambda^2)} \right) \frac{t^\beta}{\Gamma(1+\beta)}$ and c_1 and c_2 are arbitrary constants.

Some special cases:

Case I: By putting $c_1 = 0$, it yields

$$u_5 = \frac{3b(4v-\lambda^2)}{2a(4vb+1-b\lambda^2)} \left(1 - \coth^2 \left(\frac{\sqrt{\lambda^2-4v}}{2\Gamma(1+\alpha)} x^\alpha - \frac{\sqrt{\lambda^2-4v}}{2(4vb+1-b\lambda^2)\Gamma(1+\beta)} t^\beta \right) \right)$$

and for setting $c_2 = 0$, we get

$$u_6 = \frac{3b(4v-\lambda^2)}{2a(4vb+1-b\lambda^2)} \left(1 - \tanh^2 \left(\frac{\sqrt{\lambda^2-4v}}{2\Gamma(1+\alpha)} x^\alpha - \frac{\sqrt{\lambda^2-4v}}{2(4vb+1-b\lambda^2)\Gamma(1+\beta)} t^\beta \right) \right)$$

Case II: Similarly for $c_1 = 0$, it yields

$$u_7 = \frac{3b(4v-\lambda^2)}{2a(4vb+1-b\lambda^2)} \left(1 + \cot^2 \left(\frac{\sqrt{4v-\lambda^2}}{2\Gamma(1+\alpha)} x^\alpha - \frac{\sqrt{4v-\lambda^2}}{2(4vb+1-b\lambda^2)\Gamma(1+\beta)} t^\beta \right) \right)$$

and if $c_2 = 0$, then

$$u_8 = \frac{3b(4v-\lambda^2)}{2a(4vb+1-b\lambda^2)} \left(1 + \tan^2 \left(\frac{\sqrt{4v-\lambda^2}}{2\Gamma(1+\alpha)} x^\alpha - \frac{\sqrt{4v-\lambda^2}}{2(4vb+1-b\lambda^2)\Gamma(1+\beta)} t^\beta \right) \right)$$

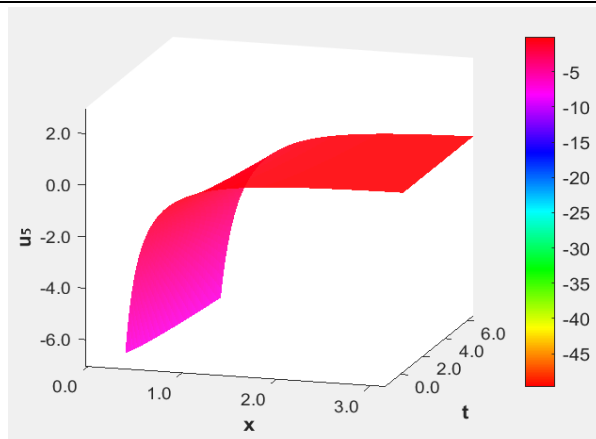


fig5: $\alpha = 0.6, \beta = 0.5, \lambda = 1.0, v = 1.5, a = 1.0, b = 1.5$

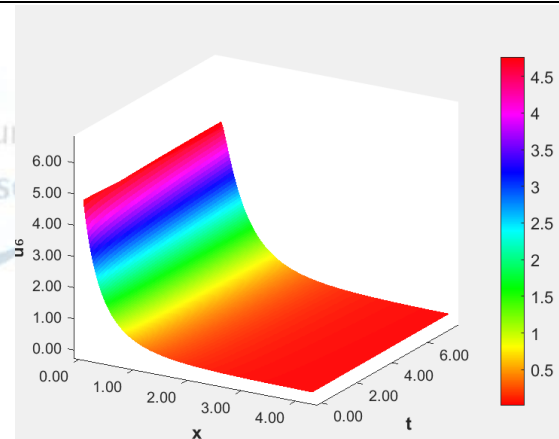


fig6: $\alpha = 0.5, \beta = 0.5, \lambda = 2.0, v = 3.0, a = 2.0, b = 3.0$

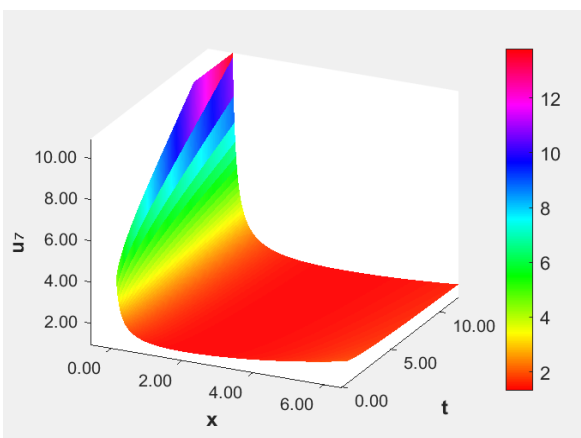


fig7: $\alpha = 0.3, \beta = 0.3, \lambda = 1.2, v = 2.2, a = 1.0, b = 0.7$

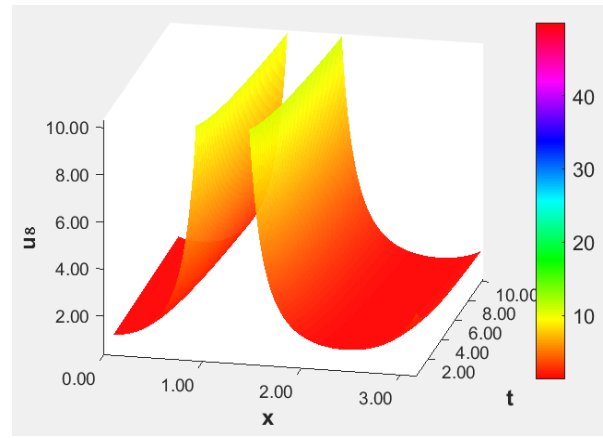


fig8: $\alpha = 0.9, \beta = 0.3, \lambda = 1.0, v = 2.3, a = 1.0, b = 4.0$

5. Numerical Results and Discussion

Graphical representation is a high-powered tool for report and tell of the solutions of problems logically. As a result, for different values of parameters are given to obtain the graph. These solutions are soliton and periodic of space-time fractional differential equations. Closed-form solutions describe different nonlinear waves result. If the speed is positive, the soliton wave moves in right-hand direction and if the speed is not positive, it displaces in left-hand direction. By changing of various parameters, the speed and amplitude both can be controlled. Now, let look at Figures 1-4 with illustrations some of our exact solutions named Dark soliton, bright soliton, Kink-like transition and singular soliton respectively achieved from the Klein-Gardner equation of selecting the some particular values of parameters and Figures 5-8 as it illustrates some of our exact solutions like Bright soliton periodic and singular soliton respectively obtained from the Zakharov-Kuznetsov Benjamin Bona Mahoney (ZK-BBM) equation of selecting the some particular values of parameters. These various solitons are not only mathematical concepts; they have practical applications in engineering and physics. Bright solitons are the workhorses of contemporary telecommunications; they resemble a single hump of energy. They are used in fiber optic cables to send information over thousands of kilometers without the signal degrading, and they also help generate ultra-wideband signals for wireless communication. Kink solitons, which make a smooth step between two states, are significant in materials science for describing how protons flow in certain crystals, which connects to the electrical properties of materials like ferroelectrics. Dark solitons, appearing as a short dip in a wave, are being explored for their potential in fiber optics and for comprehending complex wave behavior in things like ocean waves and water channels. singular solitons, which have an infinite spike, are more of a mathematical tool used to describe extreme and powerful phenomena, such as the creation of destructive tsunami waves in the ocean and Finally, periodic solitons, which generate a repeating pattern, assist scientists comprehend wave behavior in

systems with boundaries, like waves going around a circular path or in specific types of superconducting electronics. The results were derived using the $\left(\frac{G'}{G}\right)$ -expansion method. Graphs are represented clearly demonstrating the accuracy of this mentioned approach.

6. Conclusions

In this manuscript, we solved two different NSTFDE's which arises in wave phenomena named as fractional Klein-Gardner equation (KGE) and fractional Zakharov-Kuznetsov Benjamin Bona Mahoney (ZK-BBM) equation these equations plays very important role in many fields of engineering and science. Both equations are rewritten as Jumarie modified Riemann-Liouville in terms of the well-known derivative operator sense. The precise soliton solutions have been obtained as rational, hyperbolic, and trigonometric functions by using $\left(\frac{G'}{G}\right)$ -expansion method. The range of accurate soliton solutions found in this work are genuinely novel and revolutionary, and they may be applied in various disciplines and will greatly aid in comprehending the physical behavior of these models. The significance of the $\left(\frac{G'}{G}\right)$ -expansion method is that it is a quick, efficient, straightforward, and powerful method for solving nonlinear equations of mathematical models. It can be applied directly to any kind of problem and does not require any kind of linearization, needless assumptions, or perturbations that could alter the problem's physical behavior.

Abbreviations

The following abbreviation are used in this manuscript:

PDE	Partial differential equation
DE	Differential equation
ODE	Ordinary differential equation
LODE	Linear ordinary differential equation
KGE	Klein-Gardner equation
ZK-BBM	Zakharov-Kuznetsov Benjamin Bona Mahoney
NSTFDE	Nonlinear space-time fractional differential equation

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